A Riemannian geometry theory of human movement: The geodesic synergy hypothesis

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Abstract
Mass-inertia loads on muscles change with posture and with changing mechanical interactions between the body and the environment. The nervous system must anticipate changing mass-inertia loads, especially during fast multi-joint coordinated movements. Riemannian geometry provides a mathematical framework for movement planning that takes these inertial interactions into account. To demonstrate this we introduce the controlled (vs. biomechanical) degrees of freedom of the body as the coordinate system for a configuration space with movements represented as trajectories. This space is not Euclidean. It is endowed at each point with a metric equal to the mass-inertia matrix of the body in that configuration. This warps the space to become Riemannian with curvature at each point determined by the differentials of the mass-inertia at that point. This curvature takes nonlinear mass-inertia interactions into account with lengths, velocities, accelerations and directions of movement trajectories all differing from those in Euclidean space. For newcomers to Riemannian geometry we develop the intuitive groundwork for a Riemannian field theory of human movement encompassing the entire body moving in gravity and in mechanical interaction with the environment. In particular we present a geodesic synergy hypothesis concerning planning of multi-joint coordinated movements to achieve goals with minimal muscular effort.

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1. Introduction

Many studies of human movement have focused on kinematic properties but fewer studies have dealt with dynamics due to the lack of mathematical tools for analyzing the complex nonlinear dynamical interactions between the biomechanical degrees of freedom (DOFs) of the human body (Sekimoto, Arimoto, Kawamura, & Bae, 2008). Gravitational and mass-inertia loads on muscles change with posture and with mechanical interactions between the body and the environment. These changes greatly complicate the dynamical relationships between forces generated by muscles and resulting body movements. Even rotation about a single joint requires a coordinated activation of many muscles throughout the body to compensate for these dynamical interactions and to prevent movement about other DOFs. Inertial, centrifugal, and coriolis reaction forces have to be anticipated by the nervous system, especially during fast multi-joint movements.

Further complicating the picture is the fact that most functional multi-joint movements require the changing joint-angles to be appropriately coordinated to achieve performance goal(s). To produce this coordination the nervous system has to
couple changing joint-angles together by introducing nonlinear dynamical constraining relationships between them. We refer to such a set of constraining relationships as a movement synergy. Because there are many more joint-angles than performance variables, many different combinations of joint-angles and different coordination of those joint-angles can be employed to achieve the same performance goal(s).

How does the nervous system determine which particular coordination of joint-angles to use? One possibility is that it selects the coordination (i.e., movement synergy) that achieves performance goal(s) with minimum demand for muscular effort. It makes sense from an evolutionary point of view that the nervous system might have evolved strategies to achieve goals, such as catching prey and escaping predators, with minimum demand for muscular effort (O’Dwyer & Neilson, 2000). Moreover, everyday movements appear to be constrained by the imperative to optimize metabolic economy (see Sparrow & Newell, 1998 for review). Any computational theory of motor control that seeks to explain how such a movement synergy might be implemented must take into account the multi-linked nonlinear dynamics of the human body in interaction with the environment. As indicated earlier, this raises the question of what mathematical tools and computational techniques are required to handle such nonlinear dynamics.

We contend that Riemannian geometry provides the most appropriate mathematical framework for developing computational models of multi-joint human movement. Riemannian geometry represents motion in a nonlinear (curved) space known as a Riemannian manifold. The geometry of this curved space differs considerably from the intuitively well-understood geometry of linear Euclidean space. For example, the notion that parallel lines never meet is a Euclidean idea that does not hold in curved Riemannian space. Concepts concerning distances, areas, volumes, straight lines, angles between lines, velocities, and accelerations derived from Euclidean geometry all have to be modified. As has been shown already (Biess, Flash, & Liebermann, 2011; Biess, Liebermann, & Flash, 2007), some of the paradoxes and contradictions in existing computational theories of human movement (Hermens & Gielen, 2004) based on linear Euclidean notions can be resolved when reformulated in Riemannian space. Furthermore, as pointed out by Biess (2013), an investigator’s choice of coordinates can be problematic in models of human movement. No consistent inferences can be drawn because concepts such as anisotropy and orthogonality of covariance matrices, for example, are coordinate dependent. A Riemannian geometry framework allows theories of human movement to be expressed in a coordinate independent manner because tensor equations are invariant under coordinate transformations; that is, if a tensor equation equals zero in one coordinate system then it will equal zero in all coordinate systems.

Once a Riemannian framework for human movement is established, the theorems and propositions of that geometry offer the possibility of obtaining new insights into motor behaviour. In this paper we lay the groundwork for a Riemannian model of the entire human body moving in a gravitational field in mechanical interaction with the environment. In particular, we will use Riemannian geometry to show theoretically how the nervous system can coordinate movements to achieve goals with minimum demand by muscles for metabolic energy. We call this the geodesic synergy hypothesis.

2. Background

2.1. The human body as a multi-linked mechanical system

From a biomechanical point of view the human body is a multi-linked mechanical system with multiple biomechanical DOFs. Sekimoto et al. (2008) demonstrated that the inertia-induced movement of a multi-linked mechanical system is characterized by geodesic curves in a Riemannian manifold. Similarly, Bullo and Lewis (2005) showed that motion of multi-linked mechanical systems can be treated using Riemannian geometry and that motion in the absence of an external force corresponds to geodesic trajectories in the Riemannian manifold. Arimoto (2010) showed that the space spanned by the DOFs of multi-joint robots can be regarded as a Riemannian manifold with a Riemannian metric equal to the robot’s mass-inertia matrix. As will be developed in Section 3, the curvature of a Riemannian manifold for a multi-linked mechanical system is attributable to the changing Riemannian metric (i.e., changing mass-inertia matrix).

2.2. Previous applications of Riemannian geometry in studies of human movement

Handzel and Flash (1999) were early to present the case for using geometric methods in the study of human motor control. They pointed out that the spaces of motor DOFs had not so far been dealt with in a satisfactory way. All too often, they argued, motor DOFs were treated as a collection of a priori unrelated variables represented as a linear Euclidean space with the nonlinear geometric structure inherent in multi-linked systems ignored. Consequently, measurements and computations of lengths of paths were distorted leading to erroneous conclusions. These limitations are overcome, they explained, by employing spaces that do not have a linear structure. In general, these nonlinear spaces are represented by curved differentiable (smooth) manifolds, the basic objects of Riemannian geometry.

1 We consider minimizing muscular effort and minimizing demand by muscles for metabolic energy to be the same as minimizing the net tensions developed by muscles. This is distinct from sense of muscular effort (an aspect of kinesthesia) which correlates with the amount of neural drive to the muscles (Gandevia, 1987; McCloskey, 1981; O’Dwyer & Neilson, 2000).
Handzel and Flash (1996, 1997) initially applied Riemannian geometry to consideration of the well-known constrained rotations of the eyes. With similar geometric constraints applying equally well to joint rotations (Gielen, Vrijenhoek, Flash, & Neggers, 1997) they then introduced the same approach to hand and limb movements (Handzel & Flash, 1999). Subsequent work applying Riemannian geometry to the eye and/or head-eye system includes that of Ghosh and colleagues (Ghosh, Meegaskumbura, & Ekanayake, 2009; Ghosh & Wijayasinghe, 2010; Polpitiya, Dayawansa, Martin, & Ghosh, 2007; Polpitiya, Ghosh, Martin, & Dayawansa, 2004). Likewise, Biess (2013), Biess et al. (2007, 2011), and Arimoto and colleagues (Arimoto, 2010; Arimoto, Yoshida, Sekimoto, & Tahara, 2009; Sekimoto et al., 2008; Sekimoto, Arimoto, Prilutsky, Isaka, & Kawamura, 2009) have used Riemannian methods to study the motion of multi-linked robot systems and/or human multi-joint arm movements, and/or the swing phase of human walking. Ivancevic (2009) and Ivancevic and Ivancevic (2007, 2010, 2011) developed a Riemannian geometric framework to determine the human musculoskeletal system that includes translations and rotations at all the main synovial joints of the body including the spine modelled as a chain of flexibly coupled rigid bodies. Their model has 270 active DOFs. Similarly, Datas, Chiron, and Fourquet (2010) and Datas, Fourquet, and Chiron (2011) used a virtual human kinematic structure with 24 DOFs including rotations of the head, clavicles, shoulders, elbows, wrists, and bending of the vertebral column to simulate reaching movements to compare with human data.

All the above studies used Riemannian geometry to compute minimum length pathways on Riemannian manifolds and/or submanifolds. In each case these correspond to the geodesic trajectories of a Riemannian manifold defined by a coordinate system based on DOFs. Some studies obtained the geodesics by first computing the moment of inertia matrices (or mass-inertia matrices) of the system to define the Riemannian metric on the manifold whereas others used the special orthogonal group of rotations \(SO(3)\) to define the manifold and obtain the geodesics. In all cases comparisons between the computed geodesic pathways and actual human movements show a close match leading to the conclusion that humans use inertial properties of the body efficiently when planning and executing smooth movements.

Biess et al. (2007) used Riemannian geometry to show both theoretically and experimentally that existing incompatible computational theories of pointing movements (e.g., minimum-jerk change, minimum-torque change, minimum peak kinetic energy) are reconciled by regarding joint-angle space as a Riemannian manifold with an inertia-matrix metric and by using the methods of Riemannian geometry to compute geodesic trajectories. Subsequently they extended the minimum-jerk change and minimum-torque change models from Euclidean to Riemannian spaces and claimed that these are mathematically equivalent when formulated in a Riemannian manifold with a kinetic energy (mass-inertia matrix) metric (Biess et al., 2011).

Importantly, Biess et al. (2007) showed that geodesic trajectories minimize demand for muscular effort. Also, importantly, they drew a distinction between the spatial and the temporal planning of a response. Using a model of the arm in a four-dimensional manifold with coordinates corresponding to three rotations at the shoulder and one rotation at the elbow, they computed the unique geodesic pathway for a pointing movement connecting required initial and final configurations. They referred to this as planning the spatial part of the task. Then they introduced an ingenious idea. They used minimization of jerk-change to compute accelerations and decelerations along the geodesic pathway and referred to this as planning the temporal part of the task. Separating spatial and temporal planning in this way greatly simplifies the computational task of movement planning. The technique has since been employed in robotics by Sekimoto et al. (2009).

### 2.3. Elemental movements versus biomechanical DOFs

All the studies mentioned in the previous section defined DOFs in terms of rotations of the eye, head and/or individual synovial joints. Ivancevic and Ivancevic (2007, 2010, 2011) argued that every synovial joint involves not only rotations but also micro-translations. They claimed consequently that human motion is rigorously defined in terms of the rotations and translations of all the main human joints. True, these rotations and translations define the DOFs of the biomechanical part of the human motor system, hence we refer to them here as biomechanical DOFs. However, if we consider only movements that can be made voluntarily, these individual joint rotations and translations no longer qualify as DOFs. The anatomical arrangements of muscles and fundamental cross-couplings within the nervous system prevent voluntary control of certain biomechanical rotations and translations independently of each other. For example, in the neural control of eye movement smooth pursuit and saccadic systems involve coupling that produces conjugate movements of the eyes. In contrast, the vergence system produces disconjugate movements of the eyes. Nor is it possible to control the rotations and translations of joints in the vertebral column independently, or likewise particular joints of the fingers. In other words, with respect to voluntary control of movement, anatomical and neurophysiological constraints prevent many biomechanical DOFs from operating independently of each other. Moreover, many movements of the body do not involve synovial joints, for example, those of the face, eyes, tongue, larynx, translations and rotations of the scapula, sphincter contractions, and so forth.

In light of the above, we define the coordinate system spanning the space of voluntary movements to be based not on biomechanical DOFs but on elemental movements; the simplest movements that can be made voluntarily independently of each other one at a time (Neilson & Neilson, 2005a, 2005b, 2010). Some elemental movements involve rotation about a single axis at a single joint (e.g., flexion/extension of the first joint of the index finger). Others involve translations of body parts (e.g., raising and lowering the larynx and translations of the scapula). Again, others involve correlated (coupled) combinations of rotations and translations (e.g., movements of the face, bending of the vertebral column, mechanically coupled rotations of the second and third joints of the index finger, and rotations of synovial joints where the centre of rotation
changes with joint angle). When an elemental movement is constrained in isometric tasks, the net tension about that elemental movement remains under voluntary control. This gives rise to the notion of *elemental tensions*. Elemental movements and elemental tensions can be thought of as the (long-term) independent components of voluntary body movements averaged over long time windows to remove short-term response-dependent correlations.

### 2.4. Adaptive Model Theory and movement synergies

In Adaptive Model Theory (AMT) (Neilson & Neilson, 2005a, 2005b, 2010) we estimate that there are some 110 elemental movements in the human body. Although the order of this number is important the exact count is not crucial, so for simplicity in what follows we assume 110 to be correct. While, by definition, there are no anatomical or neurophysiological constraints preventing elemental movements from being performed voluntarily independently of each other, the central nervous system appears to have insufficient central processing resources to control all 110 elemental movements independently in parallel. We have argued that the nervous system overcomes this problem of limited central processing resources (and simultaneously solves the problem of redundancy in the musculoskeletal system) by forming response-dependent movement synergies (Neilson & Neilson, 2010). For each response, groups of elemental movements are constrained by the nervous system to move together in nonlinearly dynamically related ways. Each group is then controlled as a unit, forming a short-term independent component of the total response-dependent movement synergy and corresponding to a single response-dependent *control degree of freedom* (CDOF) of that short-duration movement. This greatly reduces the workload for central processes. Rather than planning and controlling 110 elemental movements, the central nervous system has only to plan and control a small number of CDOFs in parallel. The price for this simplified strategy of motor control is that response-dependent movement synergies have to be predetermined and switched from one response to another. This raises the question of the selection of an appropriate movement synergy for a response.

Examine the question of how the nervous system might select an appropriate response-dependent movement synergy leads quickly to the realization that the problem of redundancy has simply been shifted. An infinite number of different movement synergies are able to achieve a specified set of perceptual task goals (performance variables). One way to solve this version of the redundancy problem is to select the movement synergy that achieves the perceptual goals with minimum demand by muscles for metabolic energy. However, the question of how the nervous system can form and store in memory a repertoire of movement synergies able to achieve perceptual goals with minimum demand for muscular effort poses a computational challenge of considerable complexity. Not only must all possible synergistic voluntary movements of the entire body be allowed for but the influence of gravity and of mechanical interactions between the body and the environment must also be taken into account. Moreover, from response to response the nervous system must have the flexibility to change the performance variables (CDOFs) and the number of these required.

To overcome any uncertainty as to what we mean by ‘response-dependent movement synergy’ we have defined the notion of a *synergeme* (Neilson & Neilson, 2004). Motor behaviours, such as drinking from a glass, are comprised of a sequence of synergemes, such as reach, grasp, pick-up, transport, place against lips, tilt and swallow, tilt, transport, release grip, and so on. Synergemes are similar to but not the same as Gilbreth’s so-called *therbligs* (Ferguson, 2000). Both are useful from an experimental point of view because transitions between both synergemes and therbligs are easy to recognize. We define a *synergeme* to be a movement performed within a given movement synergy with particular sets of elemental movements constrained to move together by particular sets of nonlinear dynamical constraining relationships between them. The duration of a synergeme can be as short as a single submovement (100 ms) or it can consist of a long sequence of submovements (such as when steering a car with a fixed grip on the steering wheel). Synergemes may have only one CDOF or they may involve multiple CDOFs. As in the ‘drinking from a glass’ example above, behaviours can be comprised of sequences of synergemes with the movement synergy being switched from one synergeme to the next but, theoretically at least, sequences of short-duration synergemes each with a small number of CDOFs can be traded for a longer-duration synergeme with a larger number of CDOFs, coarticulation during speech motor control being an example.

Most skilled motor behaviours are comprised of sequences of concatenated synergemes. Each synergeme in the concatenated sequence corresponds to a different movement synergy. The nervous system has to be able to switch quickly and smoothly from one movement synergy to the next. This involves (i) selecting a movement synergy appropriate for the task (i.e., compatible with the task space) and (ii) generating the required set of constraining dynamical relationships between elemental movements (i.e., the required coordination) to achieve the task goals with minimum muscular effort taking nonlinear inertial dynamics into account. In this paper we focus only on the latter challenge. The selection problem requires geometric analysis of kinesthetic-visuospatial maps and reinforcement learning. This builds on the work set out here and will be the subject of a further paper.

### 2.5. Aims and hypothesis

In what follows we amplify the link between AMT and the notion of spatial and temporal planning introduced by Biess et al. (2007). We do so by developing the idea that the constraining relationships associated with response-dependent geodesic pathways in a Riemannian manifold correspond to the constraining relationships between elemental movements that define response-dependent geodesic movement synergies.
We generalize the model of Biess et al. (2007, 2011) in three ways and in so doing we introduce a number of novel concepts of relevance to movement control. Firstly, rather than describing subsystems of motor control, like eye/head dynamics and arm dynamics, we describe control of the entire human body. Only under contrived conditions is it possible to isolate subsystems from mechanical and neurophysiological interactions with the rest of the body and with the environment. Secondly, we expand the number of CDOFs from a single one for the pointing movements studied by Biess et al. (2007) to several, as required by most everyday multi-joint movements. This involves the introduction of the novel notion of a totally geodesic submanifold with dimension equal to the number of CDOFs. We show that selecting such a submanifold is geometrically equivalent to selecting a minimum muscular effort geodesic movement synergy for tasks with two or more CDOFs such as two-dimensional visual tracking. Thirdly, we take into account the influence of gravity as well as all mechanical interactions within the body and between the body and support surfaces in the environment. These extensions provide the theoretical basis for the geodesic synergy hypothesis:

The configuration manifold \((C, J)\) represents all the possible configurations of the human body moving in a local environment. Spatial response-planning processes generate geodesic movement synergies (coordination) that are the most efficient movement synergies for achieving performance goals with minimum demand for muscular effort. Geometrically, each such synergy corresponds to a totally geodesic submanifold with dimension equal to the number of control degrees of freedom (CDOFs) of the required response. These submanifolds are spanned by a coordinate system of geodesic grid lines embedded in the Riemannian configuration manifold \((C, J)\). Temporal response-planning processes generate goal-oriented minimum metric-acceleration movement trajectories confined to the selected totally geodesic submanifold. The resulting movement trajectories within selected geodesic synergies achieve goals with minimum muscular effort while accounting for gravitational and mechanical interactions within and between the body and its environment.

2.6. The Riemannian framework

In the two sections that follow we set out information that is key to understanding the geometrical concepts involved in the above proposal. In doing so we seek to give an intuitive appreciation of the terminology and structure of Riemannian geometry rather than a mathematically rigorous one. Detailed mathematical descriptions can be found in texts such as Abraham and Marsden (1978), Arnold (1989), Bullo and Lewis (2005), Darling (1994), Isidori (1995), Ivanovic and Ivanovic (2007, 2010, 2011), Jurdjevic (1997), Lang (1999), Lee (1997, 2011, 2013), Marsden and Ratiu, (1999), Ortega and Ratiu (2004), and Szekeres (2004). In a nutshell we can say that the Riemannian framework consists of multiple interconnected spaces or ‘manifolds’ and the relationships between them. At the heart of the framework there exists a smooth base configuration manifold (endowed with a metric) to which there are attached further vector and tensor spaces. The vectors and tensors across these spaces form fields over the base manifold analogous to electric or magnetic fields over three-dimensional space. Thus what follows can be taken as the development of a field theory of human movement.

3. The Riemannian configuration manifold

3.1. The configuration manifold requires a metric

Consider any mechanical system with \(n\) DOFs. The configuration of the mechanical system at any point \(t\) in time \(t\) can be thought of geometrically as a point \(c\) in a smooth \(n\)-dimensional configuration space \(C\) (i.e., \(c \in C\)) with coordinates \(c^1, \ldots, c^n\) corresponding to the positions of the \(n\) DOFs of the system. Any smooth change in configuration from one point \(c \in C\) to another defines a movement of the system and is represented geometrically by \(c(t)\), a smooth trajectory parametrized by time \(t\) in the configuration space \(C\).

Thus far we have made no specification about the size and shape of \(C\). We know only that it is an \(n\)-dimensional space that is smooth (i.e., infinitely differentiable). In order to have a defined size and shape, \(C\) requires a metric. If the metric were constant (i.e., the same at every \(c \in C\)) then \(C\) would be a flat Euclidean space, the \(c^1, \ldots, c^n\) would be Cartesian (rectilinear), and movement \(c(t)\) of the mechanical system would be specified linearly. In order to account for nonlinear inertial interactions between the \(n\) DOFs the metric must vary, taking differing values at each \(c \in C\) and thus ‘warping’ the configuration space from its Euclidean counterpart.

We show below that the required metric is the Riemannian-metric tensor \(J(c)\) that can be represented as an \(n \times n\) matrix \(J(c)\) where each element of the matrix \(J(c)\) corresponds to a component of the tensor. Every point \(c \in C\) is ‘endowed’ by \(J(c)\) and thus, for a nonlinear Riemannian account of the system, the configuration space is denoted by \((C, J)\). It will be seen that the Riemannian metric is the kinetic-energy metric and equivalently that the matrix \(J(c)\) is the mass-inertia matrix of the mechanical system. But to establish this, we must first set out how the base configuration space \(C\) that specifies position can be extended to include spaces that specify velocity and acceleration.

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2 The Biess model includes the effect of gravity on arm movement but this is in isolation from a whole-body context where interactive and other external forces also play a role. It does not include the influence of support forces and movement constraints imposed by support forces.
3.2. Velocity and acceleration spaces as tangent fibres and tangent bundles

The differential of a curve \( c(t) \) in \( C \) passing through \( c \in C \) is \( dc/dt \), a velocity. At the point \( c \in C \) all possible velocities associated with all possible curves passing through that point can be represented by velocity vectors \( v \) in an \( n \)-dimensional tangent velocity space \( T_v C \). Each such tangent space \( T_v C \) can be thought of as a fibre at its point \( c \in C \). In turn, the union of all the disjoint \( T_v C \) fibres can be thought of as the fibres being ‘glued together’ to form another smooth manifold \( TC \) known as the tangent bundle. Points in \( TC \) are specified by both position and velocity \((c, v)\). A vector \( v \) that varies smoothly with position \( c \in C \) from one fibre \( T_v C \) to another gives rise to a velocity vector field in bundle \( TC \) over the configuration space \( C \).

Since \( TC \) is itself a smooth space we can repeat the above reasoning and define a double tangent vector space \( T_v TC \) as a fibre at its point \((c, v) \in TC \). The differential of a curve \((c(t), v(t)) \) in \( TC \) passing through point \((c, v) \in TC \) is \((dc/dt, dv/dt)\). Thus a vector in the fibre \( T_v TC \) has two parts corresponding to velocity and acceleration. Since the velocity part is redundant (in the sense that it is isomorphic with velocity \( v \) in \( TC \)), we focus only on the acceleration part (known as the principal part) of the vector in the fibre \( T_v TC \).

The union of all the disjoint double tangent spaces \( T_v TC \) over all points \((c, v) \in TC \) forms another smooth space known as the double tangent bundle \( TTC \). Points in \( TTC \) are specified by position, velocity, velocity, and acceleration \((c, v, u, a)\). The principal part of a vector that varies smoothly with position and velocity \((c, v) \in TC \) from one fibre \( T_v TC \) to another gives rise to an acceleration vector field \( a(c, v) \) in double bundle \( TTC \) over bundle \( TC \).

3.3. The Riemannian-metric tensor space and the Riemannian metric

We now introduce the Riemannian-metric tensor space. Like the velocity vector space \( T_v C \), this space can be thought of as a fibre at each point \( c \in C \). It contains a vector-like tensor known as the Riemannian-metric tensor \( j(c) \). By definition \( j(c) \) is a symmetric, positive definite, non-singular, bilinear map that acts at the point \( c \in C \) on any two vectors \( v \) and \( z \) in the tangent space \( T_v C \) and transforms them into a real number equal to the Riemannian-metric inner product of the two vectors, i.e.,

\[
j(c)(v, z) = \langle v, z \rangle = \langle v, z \rangle_{j(c)} = \langle j(c)v, z \rangle.
\]

(1)

The tensor \( j(c) \), and hence the matrix \( j(c) \), varies smoothly from point to point in the configuration space \( C \) thereby forming a smooth Riemannian-metric tensor field \( j \), or equivalently, a smooth Riemannian-metric matrix field \( j \) over the space \( C \). The space \( C \) together with its Riemannian-metric matrix field \( j \) defines a Riemannian configuration manifold denoted by \( (C, \langle \cdot, \cdot \rangle_j) \).

When the Riemannian-metric matrix \( j(c) \) operates on a vector \( v \) it transforms it into another vector \( j(c)v \) with a different length and direction. This transformation depends on both the matrix \( j(c) \) and the direction of the vector \( v \) in \( T_v C \). The Riemannian-metric norm \( ||v||_{j(c)} \) of a vector \( v \) in \( T_v C \) (i.e., the length of the vector) is equal to the square root of the Riemannian-metric inner product, i.e., \( \langle v, v \rangle_{j(c)}^{1/2} \), and consequently, the Riemannian-metric norm of a velocity vector \( v \) depends not only on the position \( c \in (C, \langle \cdot, \cdot \rangle_j) \) in the Riemannian configuration manifold but also on the direction of \( v \) in each fibre \( T_v C \). The angle \( \theta \) between two vectors \( v \) and \( z \) in the fibre \( T_v C \) is defined using Riemannian-metric inner products as

\[
\cos \theta = \frac{\langle v, z \rangle_{j(c)}}{||v||_{j(c)}^{1/2} ||z||_{j(c)}^{1/2}}.
\]

(2)

Vectors in \( T_v C \) that have unit Riemannian-metric norms and are mutually orthogonal (i.e., at right angles to each other) will be referred to as \( J \)-orthonormal vectors (and become important in Section 5). Meanwhile, in light of the above properties, we can now establish that \( j(c) \) equates to a kinetic-energy metric.

3.4. The Riemannian metric is the kinetic-energy metric

For a mechanical system with \( n \) DOFs moving with velocity \( v \) at configuration \( c \in C \) the kinetic energy at \( c \in C \) is given by the expression \( \frac{1}{2}m \dot{v}^2 = \frac{1}{2}m(c)\langle v, v \rangle \), where \( m(c) \) is the mass-inertia tensor of the system in configuration \( c \in C \). From Section 3.3 we can see that for a Riemannian configuration manifold the kinetic energy of the system moving with velocity \( v \) through configuration \( c \in (C, \langle \cdot, \cdot \rangle_j) \) can be expressed in terms of the Riemannian-metric inner product as \( \frac{1}{2}j(c)\langle v, v \rangle \) where the Riemannian-metric tensor \( j(c) \) at every \( c \in (C, \langle \cdot, \cdot \rangle_j) \) is equal to the mass-inertia tensor \( m(c) \) of the system in every configuration \( c \in (C, \langle \cdot, \cdot \rangle_j) \), or equivalently, where the Riemannian-metric matrix \( j(c) \) in every configuration \( c \in (C, \langle \cdot, \cdot \rangle_j) \) is equal to the mass-inertia matrix of the system in every configuration \( c \in (C, \langle \cdot, \cdot \rangle_j) \). When the Riemannian-metric matrix \( j(c) \) is set equal to the mass-inertia matrix of the system it is known as the kinetic-energy metric (for an in-depth description of the relationship between kinetic energy and the Riemannian metric see Bullo and Lewis (2005)).
The Riemannian-metric matrix $J(c)$ is fundamental to Riemannian geometry because it is this that defines the metric structure of the Riemannian configuration manifold $(C, J)$. Because $J(c)$ varies with $c \in (C, J)$ it warps $(C, J)$ by changing distances between points as a function of position in the manifold. For this reason Riemannian geometry can be thought of as the geometry of an elastic space that can be stretched or compressed by any amount in any direction by the Riemannian metric $J(c)$. An alternative (and more usual) way of thinking about warping is to regard $(C, J)$ as a curved space. The curvature of $(C, J)$ at each point depends on the first and second derivatives of the metric $J(c)$ at each point. This curvature changes distances, velocities and accelerations of trajectories in $(C, J)$ compared with their counterparts in Euclidean $C$.

As we have seen, for a mechanical system with $n$ DOFs, $J(c)$ is given by the mass-inertia matrix of the system at each point $c$ in the $n$-dimensional manifold $(C, J)$. A trajectory $c(t)$ in $(C, J)$ still represents a movement of the system as in Section 3.1 but it is now endowed with a kinetic-energy metric $J(c)$ that allows path lengths and directions, as well as magnitudes and directions of velocities and accelerations, to be computed taking inertial nonlinear interactions between the $n$ DOFs into account. In other words, the Riemannian (or kinetic-energy) metric in conjunction with the $n$-dimensional configuration manifold $(C, J)$ defines the inertial nonlinear dynamics of the mechanical system.

3.5. The configuration manifold has a predetermined acceleration field

Conservation of kinetic energy ($K = \frac{1}{2}m \dot{v}^2$, $m =$ mass, $\dot{v} =$ velocity) is a basic principle of mechanics that can be used to compute the motion of any simple mechanical system in the absence of an external force (Bullo & Lewis, 2005). If the mass $m$ is constant then conservation of kinetic energy is equivalent to Newton’s first law of motion: ‘a body remains in a state of rest or uniform motion in a straight line unless acted on by an external force’. If the mass changes, however, then to conserve kinetic energy, the velocity has to change to compensate for the changing mass. In other words, the mechanical system will accelerate or decelerate to conserve kinetic energy despite the absence of an external force.

This natural positively or negatively accelerating motion of the mechanical system in the absence of external force is taken into account in Riemannian geometry within the structure of the configuration manifold $(C, J)$. It is known as the principal part or acceleration part of the geodesic spray field (Lang, 1999) stored in TTC. Since it is the second (principal) part of the geodesic spray field and it depends on both position and velocity we denote it by $f_2(c, v)$. If the mass-inertia matrix $J(c)$ is known for every configuration, the kinetic energy $K$ for every position and velocity $(c, v)$ can be computed using the metric inner product $K = \frac{1}{2}v^T J^{-1} v$, where the mass-inertia matrix $J(c)$ equals the Riemannian metric at the point $c \in (C, J)$. The spray acceleration vector $f_2(c, v)$ can then be predetermined for every position and velocity $(c, v)$ using Eq. (3) (derived from the Euler–Lagrange equation of motion in Marsden and Ratiu (1999)) and stored in the double tangent bundle $TTC$:

$$f_2(c, v) = \ddot{v} = \left[\frac{\partial^2 K}{\partial v^2}\right]^{-1} \left(\frac{\partial K}{\partial c} - \frac{\partial^2 K}{\partial c \partial v} v \right).$$

These predetermined values of the spray acceleration field $f_2(c, v)$ can be regarded as an inherent part of the Riemannian configuration manifold $(C, J)$ equal to an acceleration vector field over the tangent bundle $TC$. This is taken up later with respect to the generation of geodesic trajectories.

3.6. Naïve measures of acceleration

When it comes to computing the acceleration $\ddot{x}(t)$ along a trajectory $x(t)$ in a Riemannian configuration manifold $(C, J)$ a problem occurs that does not occur in Euclidean space. In essence, computing acceleration along a trajectory involves computing the change in velocity $x(t_1) - x(t_2)$ between two points $x(t_1)$ and $x(t_2)$ along the curve. But because a Riemannian manifold is curved the velocity vector $\dot{x}(t)$ is located in the tangent space $T_{x(t)} C$ while the velocity vector $\dot{x}(t)$ is located in a completely different tangent space $T_{\dot{x}(t)} C$. Consequently, the velocity vectors cannot be subtracted (or at least they can only be subtracted in a naïve way).

Imagine using a tape measure to measure distance. Suppose the tape measure itself expands or contracts from place to place. Measuring distances without taking the changing length of the tape measure into account would be a naïve measure of distance. A similar thing can happen when measuring acceleration $\ddot{x}(t)$ along a trajectory $x(t)$ in a Riemannian manifold. If the velocity vectors $\dot{x}(t_1)$ and $\dot{x}(t_2)$ are measured relative to the basis vectors $\partial_1, \ldots, \partial_n$ spanning the tangent vector spaces $T_{x(t_1)} C$ and $T_{x(t_2)} C$ without taking into account the fact that the tangent vector spaces and their spanning basis vectors rotate relative to each other along the trajectory because of the curvature of the manifold, then the computed $\ddot{x}(t)$ will be a naïve (Euclidean) measure of acceleration.

Einstein’s theory of gravity provides a nice illustration of naïve acceleration. If we drop a stone in a gravitational field we see it accelerate. However, according to Einstein, no force of gravity has to be hypothesized because what we see is a Euclidean acceleration attributable to the curvature of the spacetime manifold. The stone appears to accelerate because we have failed to take the curvature of the spacetime manifold into account. Similar misinterpretations occur with respect to mechanical systems if we fail to take the curvature of the configuration manifold $(C, J)$ into account. In other words, theory of the movement of such systems, the human body included, is inadequate unless it addresses the nonlinear inertial interaction forces within and between the system and the environment.
3.7. The connection and the covariant derivative on a Riemannian manifold

Computing the acceleration \( \ddot{x}(t) \) along a trajectory \( x(t) \) requires velocity vectors in disjoint tangent spaces \( T_{x(t)}C \) to be differentiated. To compute the difference between velocity vectors in two such disjoint spaces, even at infinitesimally close points along the curve, some sort of connection between the tangent spaces is required. A connection \( \nabla \) (del) is a map that transforms two vector fields in the tangent bundle \( TC \) into a third vector field in the tangent bundle. The connection \( \nabla \) on a Riemannian manifold is uniquely determined by the Riemannian-metric matrix \( J(c) \) and its first and second differentials from point to point on the manifold. This stems from the defining requirement for the connection \( \nabla \) to be compatible with the Riemannian metric \( J(c) \). The connection \( \nabla \) is said to be compatible with \( J(c) \) if it satisfies the following product rule for all vector fields \( X, Y, \) and \( Z \):

\[
\nabla_X(Y, Z)_j = (\nabla_X Y)_j + (Y, \nabla_X Z)_j.
\]

(4)

This is equivalent to the requirement for \( \nabla_j \) to be equal to zero at every point on the manifold. We return to Eq. (4) in Section 5. The connection \( \nabla \) maps any two vectors \( v \) and \( z \) in \( T_xC \) at any point \( c \in (C, J) \) into a third vector \( \nabla_v z \) in \( T_xC \) known as the covariant derivative of \( z \) in the direction \( v \). In other words, \( \nabla_v z \) at any point \( c \in (C, J) \) is a vector in \( T_xC \) equal to the directional derivative of the velocity vector \( z \) (i.e., the acceleration) in the direction \( v \). The problem is that the apparent (Euclidean) directional derivative \( \dot{z}v \) of the vector \( z \) in the direction \( v \) includes a naïve acceleration caused by the relative rotation of the basis vectors spanning \( T_xC \) associated with movement in the direction \( v \). To obtain the ‘true’ directional derivative (i.e., \( \nabla_v z \)) the naïve acceleration due to the curvature of \( (C, J) \) has to be computed and subtracted out. In other words, obtaining \( \nabla_v z \) at each point \( c \in (C, J) \) requires a measure of the naïve acceleration of \( z \) in the direction \( v \) at each point \( c \in (C, J) \) caused by relative rotation of the basis vectors spanning \( T_xC \) (i.e., due to the curvature of the manifold). This symmetrical bilinear map \( B(c; v, z) \) is given by the components of the connection (known as Christoffel symbols \( \Gamma^k_{ij} \)) that can be derived uniquely from the first and second differentials of the Riemannian-metric matrix \( J(c) \) (Lang, 1999). Thus the covariant derivative \( \nabla_v z \) is obtained by subtracting \( B(c; v, z) \) from the apparent (Euclidean) directional derivative \( \dot{z}v \), as in Eq. (5):

\[
\nabla_v z = \dot{z}v - B(c; v, z).
\]

(5)

From this it can be seen that the covariant derivative \( \nabla_{\dot{z}(t)} \dot{x}(t) \) along a trajectory \( x(t) \) is given by Eq. (6)

\[
\nabla_{\dot{z}(t)} \dot{x}(t) = \ddot{x}(t) - B(\dot{x}(t); \dot{x}(t), \dot{x}(t)),
\]

(6)

where \( \ddot{x}(t) \) is the naïve (Euclidean) acceleration along the trajectory, \( \dot{x}(t) \) is the Euclidean velocity vector tangent to the trajectory, and \( B(\dot{x}(t); \dot{x}(t), \dot{x}(t)) \) is the acceleration attributable to rotation of basis vectors of \( T_xC \) along the trajectory \( x(t) \). It is equal to the predetermined spray acceleration \( f_j(\dot{x}(t); \dot{x}(t)) \) at the point \( x(t) \) in the tangent bundle \( TC \). The point is that the naïve acceleration \( B(\dot{x}(t); \dot{x}(t), \dot{x}(t)) = f_j(\dot{x}(t); \dot{x}(t)) \) is an inherent property of the manifold \( (C, J) \) determined by the first and second differentials of the metric \( J(c) \) on the manifold, or equivalently from the differentials of kinetic energy as in Eq. (3).

3.8. Geodesic trajectories

A trajectory \( x(t) \) is defined to be a geodesic trajectory when it has a covariant derivative \( \nabla_{\dot{z}(t)} \dot{x}(t) \) at every point along the trajectory equal to zero. In other words, for a geodesic trajectory \( x(t) \), the Euclidean acceleration equals the naïve acceleration \( \ddot{x}(t) = B(\dot{x}(t); \dot{x}(t), \dot{x}(t)) = f_j(\dot{x}(t); \dot{x}(t)) \). To distinguish between the Euclidean acceleration \( \ddot{x}(t) \) along the trajectory and the ‘true’ acceleration \( \nabla_{\dot{z}(t)} \dot{x}(t) \) we refer to the covariant derivative \( \nabla_{\dot{z}(t)} \dot{x}(t) \) as the metric-acceleration. Zero metric-acceleration \( \nabla_{\dot{z}(t)} \dot{x}(t) \) along a geodesic trajectory \( x(t) \) implies that the metric-speed \( \| \dot{x}(t) \|_{\dot{z}(t)} \) along the geodesic trajectory \( x(t) \) is constant and consequently, geodesic trajectories are to Riemannian manifolds what straight lines are to Euclidean spaces. They can pass through every point in \( (C, J) \) in every possible direction, and distances along geodesic trajectories in \( (C, J) \) are the shortest distances between points taking curvature into account. The distance \( s(t) \) along \( x(t) \) in \( (C, J) \) is computed by integrating the metric-speed \( \| \dot{x}(t) \|_{\dot{z}(t)} \) along the trajectory. This takes curvature of \( (C, J) \) into account so both the speed and the distance in \( (C, J) \) differ from the Euclidean speed \( \| \dot{x}(t) \| \) and the Euclidean distance respectively, at each point \( x(t_i) \) along the trajectory. We use the terms metric-speed and metric-distance to emphasize this difference.

Geodesic trajectories are a cornerstone of the geodesic synergy hypothesis. Having set out their properties in relation to a general multi-linked mechanical system moving in a Riemannian configuration manifold \( (C, J) \) we are now ready to apply this to movement of the entire human body moving in a local environment.
4. A Riemannian formulation of the entire human body moving in a local environment

4.1. The configuration manifold

Three smooth manifolds are required to specify uniquely the configuration of the entire human body in a local environment: (i) A 110-dimensional elemental movement manifold $\Theta$, a point in $\Theta$ corresponding to body posture in terms of the positions of the 110 elemental movements (Section 2.3). (ii) A three-dimensional place manifold (place map) $P$, a point in $P$ giving the position of the head in the extrinsic three-dimensional environment relative to an external reference frame $X, Y, Z$. (iii) A three-dimensional orientation manifold $O$, a point in $O$ giving the orientation (rotation) of the head relative to the external reference frame. The location and orientation of the head in the local environment can be controlled voluntarily via changes in the positions of elemental movements, for example, by walking or lying down. It is possible, therefore, to form dynamic synergistic maps between the $\Theta, P$, and $O$ spaces. It is also possible, however, for the location and orientation of the head to be varied independently of body posture; consider for example, the changing location and orientation of the head of a ballerina as she is carried about the stage by her partner.

The Cartesian product $C = \Theta \times P \times O$ of these three smooth manifolds defines a 116-dimensional space $C$. A point $c \in C$ has coordinates $c^1, \ldots, c^{116}$ that uniquely specify body posture, head position, and head orientation relative to the $X, Y, Z$ frame of the local environment. $C$ thus defines a smooth base manifold of the human body. Any change in the configuration of the body means a change from one point $c \in C$ to another. This, of course, defines a body movement that is represented by $c(t)$, a trajectory in the manifold $C$ parametrized by time $t$.

In light of the principles established in Section 3 we can now make the 116-dimensional manifold $C$ Riemannian. By endowing $C$ with $J(c)$, where $J(c)$ is the mass-inertia matrix of the human body, $C$ becomes the 116-dimensional Riemannian manifold $(C, J, J(c))$ changes with configuration but, once it is known, the kinetic energy for every position and velocity $(c, v) \in TC$ can be computed as in Section 3.5. This in turn determines the spray acceleration vector field $f_\Sigma(c, v)$ on $TC$. Indeed, the predetermined spray acceleration vector field $f_\Sigma(c, v)$ equals the predetermined bilinear map $B(c, v)$ described in Sections 3.7 and 3.8. In the absence of all external forces (gravity, support forces, muscle forces, elastic forces, frictional forces, aerodynamic forces, etc.), the apparent free (natural) motion of the human body is to accelerate or decelerate, as determined by the naïve acceleration vector field $f_\Sigma(c, v)$. Acceleration-deceleration of a spinning ice-skater according to arm position nicely illustrates the principle. But our interest is in human movement in a gravitational field with the body supported by objects in the local environment; e.g., standing on the floor, leaning against a wall, sitting in a chair, lying on a bed, and so forth. In these circumstances there are external forces acting on the body, so how can $f_\Sigma(c, v)$ be relevant?

4.2. Gravitational and support forces influence the mass-inertia matrix

The dominant external forces acting on the body are gravitational forces, support forces, and muscle forces. For the body in a gravitational field in interaction with support surfaces to be held in a fixed (equilibrium) configuration, muscle forces are required to compensate for the gravitational and support forces. In other words, to maintain an equilibrium configuration the net external force acting on the body must be zero. We contend that so long as the gravitational, support, muscle and all other external forces acting on the body sum to zero at every point along the geodesic trajectory (i.e., the net external force is held at zero throughout the movement), the free motion of the body will follow the geodesic trajectory determined by the geodesic spray accelerations $f_\Sigma(c, v)$. We discuss this generalized equilibrium point hypothesis in more detail in Section 9.2. The important point here is the assertion that in the presence of external forces but with those external forces summing to zero (i.e., zero net external force at every point along the movement trajectory), the free motion of the body equals a geodesic trajectory in $(C, J)$. However, as explained below, the presence of gravitational and support forces effectively changes the mass-inertia matrix $J(c)$ and consequently geodesic motions in the presence of gravitational, support, and muscle forces (summing to zero) differ from the geodesic motions in the absence of external forces.

It is important to note that in the presence of a gravitational field, the distribution of support forces on the body constrains movement. For example, when standing on one leg it is not possible to make a kicking movement with that leg. To make such a kicking movement it is necessary first to change the configuration of the body, say by shifting weight to the other leg, so that support forces no longer constrain movement of the first leg. For a rolling wheel, the point of contact between the wheel and the ground (i.e., the support point on the wheel) is constrained to have zero velocity relative to the ground, assuming the wheel is not skidding. Similarly, if the body is not sliding on support surfaces, the support points on the body are constrained to have zero velocity relative to the support surfaces.

Fixed support surfaces act like infinite mass-inertia loads. No matter how much force muscles generate they are unable to accelerate support points on the body through the fixed support surfaces. This infinite resistance to movement offered by support surfaces is ‘reflected’ through the body as changes in the mass-inertia loads on muscles, e.g., consider the difference in mass-inertia load encountered by muscles acting about the ankle during the swing phase of a walking cycle compared

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3 It might be thought intuitively that muscle forces, being generated internally in the body, are internal forces. But, from a physical point of view, muscle forces act on the mass-inertia of the body causing it to accelerate or decelerate and thus are considered external.
with that during the stance phase. The moment of inertia of the foot about the ankle with the position of the leg fixed (swing phase) is smaller than the moment of inertia of the body about the ankle with the position of the foot fixed (stance phase). The moment of inertia of the body about the ankle with the foot fixed has been modelled by many as an inverted pendulum (e.g., Fitzpatrick, Gorman, Burke, & Gandevia, 1992; Fitzpatrick & McCloskey, 1994; Winter, Patla, Prince, Ishac, & Giello-Perczak, 1998). It is equal to the product of the relatively large mass of the body and the relatively large moment arm between the centre of mass of the body and the axis of rotation at the ankle. This moment arm changes, and hence the moment of inertia of the body about the ankle changes, with rotations at the knee, hip or other joints. This nicely illustrates the general principle, taken into account in the geodesic synergy hypothesis, that moments of inertia encountered by muscles acting about joints depend not only on the posture of the body but also on the distribution of support forces and reaction forces acting on the body. As the body moves, thereby changing its configuration, the distribution of its support forces and support points changes, thereby changing the effective mass-inertia loads on muscles, e.g., consider the changing distribution of support forces and support points across the body and the way this changes the mass-inertia loads on muscles associated with rolling over in bed.

These complications to the dynamics of human movement are dealt with in the Riemannian formulation. The mass-inertia matrix $J(c)$ includes the influences of constraints on movement imposed by the changing distribution of support forces and support points across the body. Despite the requirement for the net external force to be zero, gravitational and support forces modify the mass-inertia matrix $J(c)$ and the way it changes with position in the configuration manifold and therefore contribute to the predetermined geodesic spray acceleration field $f_2(c, v)$ that resides in the double tangent bundle $TTC$.

Similarly, holding a weight in the hand, or on any other part of the body, changes the mass-inertia matrix of the body and modifies the geodesic spray acceleration field. But if the body plus weight can be held in a fixed configuration then the net external force must be zero. Thus despite mechanical interactions with support surfaces and objects in the local environment, if the resulting changes in the mass-inertia matrix of the body are incorporated in the principal part of the geodesic spray field $f_2(c, v)$, with muscle forces tuned to maintain zero net external force, then the free motion of the body corresponds to a geodesic trajectory in the Riemannian configuration manifold $(C, J)$.

### 4.3. Vector fields in TC and TTC provide a geometric picture of nonlinear differential equations and their solutions

Suppose $\gamma(t)$ is a smooth curve in $(C, J)$ parametrized by time $t$. For each point along $\gamma(t)$ the velocity vector $\dot{\gamma}(t)$ is obtained by differentiating along the curve. It is a vector tangent to the curve in the tangent space $T_{\gamma(t)}C$ at each point $\gamma(t)$ along the curve. It is possible to think of this in the reverse direction: given a vector at each point in the tangent bundle $TC$ (i.e., a velocity vector field on $(C, J)$), and given an initial position $\gamma(0)$ (i.e., point in the configuration manifold), there is a unique curve $\gamma(t)$ in $(C, J)$ whose velocity at each point along the curve is equal to the given velocity vector field $\dot{\gamma}(t)$ at that point. Since going from $C$ to $TC$ involves differentiation, going in the reverse direction from $TC$ to $C$ involves integration. Consequently, the curve $\gamma(t)$ is referred to as an integral curve of the vector field $\dot{\gamma}$. A family of initial positions in $(C, J)$ gives a family of integral curves in $(C, J)$ known as the integral flow of the vector field. Thus, a vector field in $TC$ provides a geometrical representation of a first-order nonlinear differential equation, and its associated integral flow provides a geometrical representation of its solution set. Specifying an initial position $\gamma(0)$ selects a unique (particular) integral curve from the solution set.

A similar geometrical picture holds for smooth vector fields in $TTC$. The principal part of such a vector field can be thought of as an acceleration vector field. Integrating once gives an integral flow of velocities in $TC$, and integrating twice (double integration) gives an integral flow of position in $(C, J)$. Thus the principal part of a vector field in $TTC$ provides a geometrical representation of a second-order nonlinear differential equation and its associated integral flows in $TC$ and $(C, J)$ provide a geometrical representation of its solution set. Obtaining a particular solution requires specification of both an initial position $c$ in $(C, J)$ and an initial velocity $v$ in $T_{c}C$. In Section 4.4 we take advantage of this geometric picture to describe a relatively simple way to generate solutions of the nonlinear second-order dynamical equations of motion (Euler–Lagrange equations) of the human body taking the nonlinear inertial interactions within and between elemental movements and the environment into account.

### 4.4. Geodesic trajectory generator

Ability to generate geodesic trajectories $x(t)$ in the Riemannian configuration manifold $(C, J)$ plays a key role in the geodesic synergy hypothesis. Given an initial configuration and velocity $(x(0), \dot{x}(0))$, and taking advantage of the fact that the acceleration-deceleration vector field $f_2(c, v)$ is predetermined and stored in $TTC$ as an inherent part of the configuration manifold, the geodesic trajectory $(x(t), \dot{x}(t))$ of the human body in the absence of a net external force is easily computed. As described in Section 4.3, this is achieved by double integrating the available acceleration-deceleration vector field $f_2(x(t), \dot{x}(t))$ along the trajectory $x(t)$. As illustrated schematically in Fig. 1, given initial values $(x(0), \dot{x}(0))$, the geodesic trajectory can be generated quickly by a circuit corresponding to an array of double integrators with access to $f_2(x(t), \dot{x}(t))$ stored in memory.

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4. Initial value versus two-point boundary value geodesics are discussed in Section 9.3.
in computing the geodesic trajectory approach of Biess et al. (2011).

A schematic illustration of a geodesic trajectory generator (GTG). The block diagram shows how an array of 116 double integrators with feedback of the geodesic spray acceleration vector \( f(T,T) \) retrieved from the double tangent bundle TTC can be used to generate a geodesic trajectory \( (\alpha(t), \dot{\alpha}(t)) \) in the Riemannian configuration manifold \((C, J)\). The initial position \( \alpha(0) = (\alpha^1(0), \ldots, \alpha^{116}(0)) \) and the initial velocity \( \dot{\alpha}(0) = (\dot{\alpha}^1(0), \ldots, \dot{\alpha}^{116}(0)) \) are set, component by component, as initial values on the outputs of the integrators. The outputs of the integrators are sampled with sample time-interval \( \Delta t \) by zero-order-hold sampling devices represented by the boxes labelled ZOH. Rectangular pulses (with pulse widths \( \Delta t \)) equal to \( (\alpha^1(k), \ldots, \alpha^{116}(k)) \) and \( (\dot{\alpha}^1(k), \ldots, \dot{\alpha}^{116}(k)) \) at the sampled outputs of the ZOH boxes are the components of the sampled position \( \alpha(k) \) and sampled velocity \( \dot{\alpha}(k) \) at sample number \( k \). The sampled position \( \alpha(k) = (\alpha^1(k), \ldots, \alpha^{116}(k)) \) and sampled velocity \( \dot{\alpha}(k) = (\dot{\alpha}^1(k), \ldots, \dot{\alpha}^{116}(k)) \) at sample number \( k \) are connected to the double tangent bundle (association memory) TTC to access the components of the geodesic spray acceleration vector \( f(T,T) = f_1(k), \ldots, f_{116}(k) \) associated with the position and velocity \( (\alpha(k), \dot{\alpha}(k)) \). The sampled component signals \( f_1(k), \ldots, f_{116}(k) \) are connected, component by component, to the inputs of the double integrators. Operating in continuous-time, the integrators integrate their inputs over the sample-time interval \( \Delta t \). At the next sample number \( (k+1) \) the outputs of the integrators are sampled again by the ZOH sampling devices and the cycle repeats. Over time the outputs from the integrators generate a geodesic trajectory \( (\alpha(t), \dot{\alpha}(t)) \) that is stored, component by component, in working memory. The time required for the GTG to generate a geodesic trajectory can be reduced by any amount simply by increasing the gains of the integrators and reducing the sample-time interval \( \Delta t \). Allowing for the ability of recursively connected neural circuits to integrate their synaptic inputs, for the bursting activity of cortical columns that resemble (approximately) rectangular pulses, and for the existence of association memory networks in the cerebral cortex, it does not seem unreasonable to suggest that the GTG described can be implemented by neural circuits within the brain.

The geodesic trajectory \( \alpha(t) \) that results from this computation corresponds to an accelerating-decelerating trajectory in Euclidean space but, when curvature is taken into account, it appears as a constant metric-speed geodesic trajectory with zero metric-acceleration in the Riemannian configuration manifold \((C, J)\). Almost all the computational workload involved in computing the geodesic trajectory \( \alpha(t) \) is contained in the predetermined principal part of the geodesic spray field \( f_2(c, v) \) stored in the double tangent bundle TTC. Both the temporal and spatial parts of the geodesic trajectory are determined by double integrating the acceleration field \( f_2(c, v) \). We refer to the double integrating circuit of Fig. 1 as a geodesic trajectory generator (GTG). Simulation of a GTG is described in Section 8 below.

4.5. Spatial and temporal independence along a geodesic pathway

We established in Section 3.7 that the covariant derivative (metric-acceleration) \( \nabla_{\dot{\alpha}} \dot{\alpha}(t) \) along a geodesic trajectory \( \alpha(t) \) is zero and consequently, as indicated above, the metric-speed \( \|\dot{\alpha}(t)\|_{\gamma(t)} \) is constant along the trajectory. Importantly,

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5 The GTG provides a rapidly implementable solution of an initial value problem. In Section 9.3 this is contrasted with the two-point boundary value approach of Biess et al. (2011).
however, the path of the trajectory and the speed at which it is traversed, are independent. The constant metric-speed can be set higher or lower, depending simply on the initial velocity vector $\dot{a}(0)$, yet the geodesic path remains the same. Of course, if the metric-speed along the path is varied, this requires application of a tangential metric-acceleration-deceleration, in which case the trajectory is no longer geodesic, yet the pathway remains unchanged. We can therefore refer to the geodesic pathway $\dot{x}(s)$ as distinct from the geodesic trajectory $x(t)$. And since $s$ measures the metric-distance along geodesic trajectory $x(t)$ (i.e., arc length) we can denote the temporal flow along $x(s)$ by $s(t)$.

As recognized by Biess et al. (2007) in their movement study, the fact that the geodesic pathway is independent of the temporal flow along it is of key importance in the geodesic synergy hypothesis. It is this that allows separation of the spatial and temporal aspects of response planning.

We are now ready to develop the geodesic synergy hypothesis presenting spatial and temporal formulations in Sections 5 and 6, respectively.

5. Spatial aspects of response planning

5.1. Equivalence of spatial planning and synergy selection

To achieve a pointing movement goal, Biess et al. (2007) computed a geodesic pathway to connect specified initial and final postures of the arm in a four-dimensional Riemannian manifold spanned by three rotations at the shoulder and one rotation at the elbow. They referred to this as planning the spatial aspect of the task. Specifying a geodesic pathway is equivalent to specifying a set of nonlinear dynamical constraining relationships between the four joint angles. Imagine a point moving back and forth along the geodesic pathway in the four-dimensional joint-angle space. The moving point projects on to each of the four joint-angle coordinates generating four temporal joint-angle signals. Because the point is constrained to move along the one-dimensional geodesic pathway it follows that the four joint-angle signals have to be perfectly related in a nonlinear dynamical way. According to our definition of ‘movement synergy’ (Section 2.4) such a set of nonlinear dynamical constraining relationships defines a movement synergy for the arm. Thus the spatial aspect of response planning described by Biess et al. (i.e., generating an appropriate geodesic pathway to connect two points in joint-angle space) can be seen as equivalent to selection of an appropriate one-CDOF geodesic movement synergy to generate the reaching response from the initial joint-angle position.

This equivalence between spatial planning and selection of a geodesic movement synergy can be generalized to movements of the entire human body and to tasks requiring more than one CDOF. Flow along a geodesic pathway in the 116-dimensional configuration manifold $(C, J)$ projects on to the coordinate axes $c^1, \ldots, c^{116}$ generating a set of 116 coordinate temporal signals $c^1(t), \ldots, c^{116}(t)$. Because the geodesic pathway is one-dimensional it follows that the changing coordinate signals have to be perfectly related to each other in a nonlinear dynamical way. As described in Section 2.4, such a set of nonlinear dynamical constraining relationships between coordinate temporal signals defines a one-CDOF geodesic movement synergy involving the entire body.

Importantly, it is not the temporal waveforms of the coordinate signals per se that determine the synergy but rather the constraining relationships between them. For example, suppose a point $s(t)$ moves back and forth along a geodesic pathway $x(s)$ in $(C, J)$ parametrized by metric-distance $s$ along the pathway. It makes no difference whether the point $s(t)$ moves in an irregular stochastic fashion or in a simple sinusoidal manner (or in any other temporal fashion) along the geodesic pathway, the constraining relationships between the coordinate signals remain the same. It is the pathway $x(s)$ that determines the constraining relationships and not the temporal signal $s(t)$. On the other hand, changing the geodesic pathway changes the constraining relationships and consequently changes the movement synergy.

Using the concepts developed in Sections 3 and 4 we now consider generation of geodesic synergies for one-CDOF, two-CDOF, and finally $N$-CDOF responses. We then discuss the notion that geodesic synergies minimize demand for muscular effort.

5.2. Generating a one-CDOF geodesic movement synergy

Given an initial position $c = x(0)$ in $(C, J)$ and an initial velocity $e = \dot{x}(0)$ equal to a unit metric-length tangent velocity vector $e$ at time $t = 0$, it is simple, using the GTG described in Section 4.4, to generate a unit metric-speed geodesic trajectory $x(t)$. See Section 8 for simulation of the GTG. Recall from Section 4.4 that both the temporal and spatial parts of this geodesic trajectory are determined by the GTG. Also recall from Section 3.8 that the metric-speed along the geodesic trajectory is constant even though the apparent (naïve) velocity changes in both magnitude and direction. In the case given, where the initial velocity vector $e$ was chosen to have unit metric-length, the geodesic trajectory has a constant unit metric-speed. But since for unit metric-speed $t = s$, the unit metric-speed geodesic trajectory is parametrized by metric-distance (arc length) $s$ along the geodesic pathway. Remember, it is the geodesic pathway $x(s)$ that determines the one-CDOF geodesic movement synergy, not the temporal flow $s(t)$ along the pathway. Thus, within the Riemannian framework, using the predetermined spray...
acceleration-deceleration vector field $f_j(c, v)$ stored in TTC in conjunction with initial conditions $(c, e)$ mentioned above and the GTG of Fig. 1, it is straightforward to generate a geodesic pathway $\alpha(s)$, i.e., to generate a one-CDOF geodesic movement synergy. As established in Section 4.4, initial conditions $c = \alpha(0)$ and $e = \dot{\alpha}(0)$ uniquely determine the geodesic trajectory $\alpha(t)$ and hence in this case the geodesic pathway $\alpha(s)$. In other words, a particular one-CDOF geodesic movement synergy is uniquely specified by the particular initial values assigned to $c$ and $e$. But for a one-CDOF movement to fulfill its required purpose in a task, $c$ and $e$ must be chosen such that the pathway $\alpha(s)$ consists of points corresponding to configurations of the body that are compatible with the execution of that task. For example, if the task is to move a finger tip in a direct path between two marks on a sheet of paper, or to point at a target moving continuously in a stochastic fashion in one dimension on a display screen, the body must be positioned so that the finger can be in contact with the paper or the display screen throughout the required traverse. We therefore use the notation $(c, e)$ to specify a one-CDOF geodesic movement synergy with the proviso that, for a movement within that synergy, the chosen values of $c$ and $e$ must be compatible with the required task space.

5.3. Generating a two-CDOF geodesic movement synergy

Many functional movements require the simultaneous control of two CDOFs, for example, moving a finger tip to trace the character 3 drawn on a sheet of paper, or to track a target moving stochastically in two dimensions on a display screen. A two-CDOF geodesic movement synergy for a two-DOF arm is simulated in Section 8 below.

A two-CDOF movement synergy requires the elemental movements of the body to be coordinated in such a way that two (short-term) independent components of the resulting movement (say, the $x$ and $y$ movements of the finger on the paper or display screen) can be controlled simultaneously and independently. This is equivalent to the generation of a set of nonlinear dynamical constraining relationships between the coordinate temporal signals $c^1(t), \ldots, c^{116}(t)$ that constrain movement trajectories to remain within a two-dimensional submanifold embedded in $(C, J)$ and spanned by two coordinate axes corresponding to the two CDOFs of the movement. The two independent components of the movement can then be generated by independent movement trajectories along each of the coordinate axes, establishing a two-CDOF movement synergy. But there is more to it than this.

To minimize muscular effort the synergy has to be a geodesic movement synergy, a sort of two-dimensional equivalent of a geodesic pathway. The two-dimensional submanifold must be such that any two arbitrarily chosen points in the submanifold can be connected by a geodesic trajectory contained entirely within the submanifold. Evidently, generating a two-CDOF geodesic movement synergy is equivalent, geometrically, to generating a geodesic coordinate system for a two-dimensional submanifold centred about a specified point $c \in (C, J)$ embedded in the configuration manifold $(C, J)$ such that not only the coordinate axes but also all of the coordinate grid lines are geodesic pathways. Any submanifold with these required properties is known as a totally geodesic submanifold. The detail of generating such a submanifold requires consideration of some additional properties of Riemannian geometry. To preserve continuity of the main exposition these are set out in Appendix A. With reference to that Appendix we set out here only the basic principles involved.

Construction of a two-dimensional totally geodesic submanifold about a configuration $c \in (C, J)$ requires the specification of two $J$-orthonormal vectors $e_1$ and $e_2$ in the tangent space $T_c C$ (satisfying conditions given in Appendix A). The position and velocity $(c, e_1)$ are used as initial conditions in the GTG (Fig. 1) to generate a local unit metric-speed geodesic pathway $\alpha(x^1)$ parametrized by metric-distance (arc length) $x^1$ along the pathway measured from $c \in (C, J)$. Similarly, the position and velocity $(c, e_2)$ are used as initial conditions in the GTG to generate a second unit metric-speed geodesic pathway $\beta(x^2)$ emanating from point $c \in (C, J)$ in a direction orthogonal to $e_1$, parametrized by $x^2$ along the pathway measured from $c \in (C, J)$. Locally, at least, the geodesic pathways $\alpha(x^1)$ and $\beta(x^2)$ form geodesic coordinate axes spanning the two-dimensional totally geodesic submanifold $S_{\text{II}}$ (Appendix A).

The $J$-orthonormal vectors $e_1$ and $e_2$ can be parallel translated (Appendix A) along both the geodesic coordinate axes $\alpha(x^1)$ and $\beta(x^2)$. At every point $x^1$ along $\beta(x^2)$ the parallel translated vector $Pe_2$ remains tangent to $\beta(x^2)$ and the parallel translated vector $Pe_1$ remains $J$-orthonormal to $Pe_2$. Of course, each point $x^2$ along $\beta(x^2)$ corresponds to a point $c^2 \in (C, J)$. The position and velocity $(c_2, Pe_1)$ at each $x^2$ along $\beta(x^2)$ can be used as initial conditions in the GTG to generate a family of horizontal geodesic coordinate grid lines on the submanifold $S_{\text{II}}$. Similarly, by parallel translating the $J$-orthonormal vectors $e_1$ and $e_2$ along the geodesic coordinate axis $\alpha(x^1)$, initial conditions $(c^1, Pe_2)$ at each $x^1$ along $\alpha(x^1)$ can be used in the GTG to generate a family of vertical geodesic coordinate grid lines on the submanifold $S_{\text{II}}$, as illustrated in Figs. A3 and in Section 8. The submanifold $S_{\text{II}}$ can be defined only locally about $c \in (C, J)$ to avoid neighbouring geodesics crossing each other. Using these horizontal and vertical geodesic coordinate grid lines on the two-dimensional totally geodesic submanifold $S_{\text{II}}$ any point $p = (x^1, x^2)$ and velocity vector $v$ tangent to $S_{\text{II}}$ at the point $p$ can be projected (i.e., $v$ is parallel translated) along the geodesic coordinate grid lines on to the geodesic coordinate axes $\alpha(x^1)$ and $\beta(x^2)$. Consequently, any movement trajectory confined to $S_{\text{II}}$ can be decomposed into its two CDOFs by projection on to the two geodesic coordinate axes $\alpha(x^1)$ and $\beta(x^2)$ and conversely, any movement trajectory with two CDOFs contained in $S_{\text{II}}$ can be constructed by independently generating trajectories along the two geodesic coordinate axes $\beta(x^2)$ and $\alpha(x^1)$ and then combining them by projecting back along the horizontal and vertical geodesic coordinate grid lines respectively (the procedure is illustrated in Section 7.2). Notice that the entire 116-dimensional Riemannian configuration manifold $(C, J)$ with its inherent geodesic spray acceleration field
The only way for a movement to follow a geodesic pathway when gravitational and support forces are acting on the body is if it is a geodesic one) in a gravitational field muscles have to generate forces to overcome gravity. This demands metabolic energy!

Gravity acts on every segment of the human body generating translational forces and/or rotational torques about the elementary movements and requiring support forces to prevent the body from falling. To follow a preplanned trajectory (even a free motion of the human body in the absence of an external force, it would seem that no muscle force is needed to move the manifold along a geodesic trajectory contained entirely within the submanifold. Because geodesic trajectories correspond to the free motion of the human body in the absence of an external force, it would seem that no muscle force is needed to move at constant metric-speed between any two points in a totally geodesic submanifold. This is not the case for any other type of movement synergy. For any other submanifold that is not totally geodesic, the free-motion trajectory will leave the submanifold and muscle force will be needed to prevent the trajectory from leaving the submanifold. It is tempting to conclude from this that demand for muscular effort is minimal for geodesic movement synergies corresponding to totally geodesic submanifolds embedded in the configuration manifold \((C,F)\). There are, however, two complications standing in the way of reaching this conclusion. The first is the fact that everyday movements take place in a gravitational field with its associated support forces, and the second is the requirement for most functional movements to accelerate and decelerate. Let us discuss each of these in turn.

Gravity acts on every segment of the human body generating translational forces and/or rotational torques about the elementary movements and requiring support forces to prevent the body from falling. To follow a preplanned trajectory (even a geodesic one) in a gravitational field muscles have to generate forces to overcome gravity. This demands metabolic energy!

The number of CDOFs that can be controlled simultaneously is small, limited by the availability of neural processing resources within the central nervous system (Neilson & Neilson, 2005a, 2005b). While the maximum number of CDOFs is not established, thinking of a one-man band suggests it is greater than two (Oytam, Neilson, & O’Dwyer, 2005). Furthermore, in many tasks the number of CDOFs under simultaneous control will change from one synergeme (Section 2.4A) to the next. The nervous system therefore has to be able to generate geodesic movement synergies with a small but variable number \(N\) of CDOFs and be able to switch quickly and smoothly between them.

Geometrically, geodesic movement synergies with \(N\) CDOFs are equivalent to \(N\)-dimensional totally geodesic submanifolds \(S_N\) embedded in the configuration manifold \((C,F)\). Movement trajectories confined to such an \(N\)-dimensional submanifold correspond to movements with a set of nonlinear dynamical constraining relationships between the temporal coordinate signals \(c^1(t), \ldots, c^N(t)\) that allow \(N\) independent components of movement to be controlled (short-term) independently simultaneously. The procedures of Section 5.3 and Appendix A for constructing a local two-dimensional totally geodesic submanifold extend readily to the construction of a local \(N\)-dimensional totally geodesic submanifold as follows. Initial conditions \((c, e_1), \ldots, (c, e_N)\) uniquely determine a local \(N\)-dimensional submanifold \(S_N\) that is totally geodesic if and only if the \(J\)-orthonormal vectors \(e_1, \ldots, e_N\) remain tangent to \(S_N\) when parallel translated along any of the radial geodesic trajectories \(\gamma_{x_0}\) as described in Appendix A. In other words, a particular \(N\)-CDOF geodesic movement synergy is uniquely specified by the particular values chosen for \(c, e_1, \ldots, e_N\) with the above constraint imposed. But, as in the previous sections, for an \(N\)-CDOF movement to fulfill its required purpose in a task, \((c, e_1), \ldots, (c, e_N)\) must additionally be such that the local submanifold \(S_N\) contains points corresponding to configurations of the body that are compatible with the task space as described in Sections 5.2 and 5.3. We therefore use the notation \((c, e_1, \ldots, e_N)\) to specify an \(N\)-CDOF geodesic movement synergy, with the proviso that for a movement within that synergy, (i) the values of \(c, e_1, \ldots, e_N\) must be compatible with the required task space and (ii) the values of \(e_1, \ldots, e_N\) must satisfy the necessary and sufficient conditions for \(S_N\) to be a totally geodesic submanifold.

5.5. Geodesic synergies minimize muscular effort

We have seen that geodesic synergies are represented geometrically by totally geodesic submanifolds embedded in the configuration manifold \((C,F)\). We have also seen that it is possible to move between any two points in a totally geodesic submanifold along a geodesic trajectory contained entirely within the submanifold. Because geodesic trajectories correspond to the free motion of the human body in the absence of an external force, it would seem that no muscle force is needed to move at constant metric-speed between any two points in a totally geodesic submanifold. This is not the case for any other type of movement synergy. For any other submanifold that is not totally geodesic, the free-motion trajectory will leave the submanifold and muscle force will be needed to prevent the trajectory from leaving the submanifold. It is tempting to conclude from this that demand for muscular effort is minimal for geodesic movement synergies corresponding to totally geodesic submanifolds embedded in the configuration manifold \((C,F)\). There are, however, two complications standing in the way of reaching this conclusion. The first is the fact that everyday movements take place in a gravitational field with its associated support forces, and the second is the requirement for most functional movements to accelerate and decelerate. Let us discuss each of these in turn.

Gravity acts on every segment of the human body generating translational forces and/or rotational torques about the elementary movements and requiring support forces to prevent the body from falling. To follow a preplanned trajectory (even a geodesic one) in a gravitational field muscles have to generate forces to overcome gravity. This demands metabolic energy!
function of configuration is difficult. Nevertheless, it is not unreasonable that the nervous system can store in memory the efference copy and/or muscle tensions needed to maintain equilibrium configurations. In other words, through experience, a memory of net muscle forces required to compensate for gravitational and support forces able to hold the body stationary in equilibrium configurations in \((C, J)\) is acquired. The net tensions about elemental movements required to hold the body in equilibrium configurations are represented by a vector field over the configuration manifold \((C, J)\) that can be stored in the tangent bundle \(TC\). Using this memory any preplanned geodesic pathway can be followed. At each point along the pathway the net muscle tensions (or central motor commands) needed to maintain equilibrium in that configuration are retrieved from memory and applied to the muscles. This leads to a generalized equilibrium hypothesis discussed in Section 9.2.

Since the gravitational and support forces change with configuration it follows that the muscle tensions needed to compensate for them also have to change from point to point along the pathway. These gravity-compensating muscle tensions demand metabolic energy so how can movement along a geodesic pathway in the presence of gravitational and support forces minimize demand for muscular effort? The answer is that gravitational and support forces are conservative forces. The total amount of work done (i.e., energy expended) against gravity in moving an inertial load along a path between two specified points in the configuration manifold is independent of the path. Consequently, muscle forces needed to cancel gravitational and support forces play no role in determining the minimum energy pathway connecting any two specified points in the manifold. Movement at constant metric-speed along the unique geodesic pathway connecting specified points in \((C, J)\) minimizes net demand by muscles for metabolic energy despite the fact that muscle tensions must compensate for gravitational and support forces along the pathway.\(^7\) (Of course, this is not to say that other factors, such as desire to minimize demand for peak energy along the pathway, cannot come into play.)

The second complication is the fact that most functional movements are not constant metric-speed movements. They need to be metric-accelerated and/or metric-decelerated. Such accelerations and decelerations require muscle force. Thus, while a geodesic movement synergy is necessary for minimization of muscular effort, it is not sufficient. The metric-accelerations and metric-decelerations of trajectories within the totally geodesic submanifold must also be minimized. This involves temporal planning processes described in the next section.

6. Temporal aspects of response planning

6.1. Planning the temporal flow of a minimum energy movement

As noted above, most purposeful human movements are not executed at constant speed. Experimental studies have long demonstrated that the temporal course of movement trajectories usually involves acceleration-deceleration. Particularly where accuracy is concerned, an apparent single movement may consist of multiple submovements, revealing an underlying intermittency in response planning. As part of AMT we have addressed this in the form of the BUMP model, where BUMP is an acronym for Basic Unit of Motor Production. A complete account of that model and of the evidence supporting it is available in previous work (Bye & Neilson, 2008, 2010; Neilson & Neilson, 2010; Neilson, Neilson, & O’Dwyer, 1995). Here we give only a brief outline, sufficient to make clear its extension to the Riemannian framework where it provides the basis for minimization of metric acceleration-deceleration in the temporal flow along geodesic pathways.

The essentials of the BUMP formulation are: (i) Movement is planned and executed in a concatenated sequence of submovements (BUMPs). (ii) Planning and execution processes occur in a fixed temporal interval and run in parallel such that while one BUMP is being executed, the next is being planned in advance. (iii) Planning of each BUMP specifies an initial position \(x(t_i)\) and initial velocity \(v(t_i)\) at time \(t_i\) and a required final position \(x(t_f)\) and final velocity \(v(t_f)\) predicted ahead by a variable interval of time \(t_f - t_i\). (iv) BUMPs are planned so that the predicted position and velocity at the end of one becomes the initial position and velocity for the next (unless sensory feedback negates the prediction), thus providing smooth trajectories in expected circumstances. (v) The prediction interval \(t_f - t_i\) is at least the length of each BUMP, but may be much longer, the exact length varying with response strategy. (vi) The temporal course from \(t_i\) to \(t_f\) is determined by an optimal trajectory generator (OTG) that minimizes acceleration and deceleration within that planned course. (vii) There is OTG circuitry available to each CDOF of a required response and these run independently and in parallel in real-time, meaning that an \(N\)-CDOF response proceeds with no greater central time delay than does a one-CDOF response.

Experimental testing and simulation of the BUMP model, using a fixed processing interval of 100 ms, has shown that it is consistent with a variety of discrete-time and continuous-time movement phenomena including the psychological refractory period, speed-accuracy tradeoff, skilled tracking behaviour, adaptive learning of movement synergies and physiological tremor (Bye & Neilson, 2008, 2010; Oytam et al., 2005).

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\(^7\) This does not take into account effort associated with co-contraction and stretch reflex activation of muscles involved in stiffening the body and controlling visco-elastic properties. Visco-elastic forces play an important role in determining the stability and damping of body postures and movements and clearly this has an energy cost. In this paper, however, we concentrate on the dominant mass-inertia forces and leave analysis of the energy demands of postural stability and damping for a subsequent study. Nevertheless, it can be said that the gradient of gravitational forces about each elemental movement determines a type of ‘gravitational stiffness’ that changes with posture and orientation of the body. This stiffness can be represented by a vector field over the configuration manifold and consequently, stiffness control also can be dealt with within the Riemannian geometry framework.
6.2. Minimum energy temporal flow for a one-CDOF geodesic movement

A one-CDOF movement along a geodesic pathway \( x(s) \) has temporal flow designated by metric-position \( s(t) \) and metric-velocity \( \dot{s}(t) \) along the geodesic pathway. We have seen (Section 5) that pathway \( x(s) \) is provided by the GTG of Fig. 1. This minimizes muscular effort from the spatial aspect in that it specifies the minimum energy synergy of all the possible synergies relevant to the task. Likewise the temporal trajectory \( s(t), \dot{s}(t) \) is provided by the Riemannian extension of the OTG that specifies the minimum energy time course of the movement along the geodesic pathway. This is achieved by minimizing the metric-acceleration-deceleration in each of the BUMPs that constitute \( s(t), \dot{s}(t) \) along the geodesic pathway. In other words, given the inherent intermittency of movement planning and execution, the optimal movement from the temporal aspect is the concatenation of the minimum energy trajectories for the constituent BUMPs along the selected geodesic pathway.

In the Riemannian framework the OTG that provides those trajectories is structurally the same as in the Euclidean description (full details given in Bye & Neilson, 2008, 2010; Neilson et al., 1995). Computationally, the Riemannian OTG simply replaces acceleration with metric-acceleration (i.e., covariant derivative \( \nabla_{\dot{x}} x(t) \) (Section 3.7)) and similarly distances and velocities become their metric equivalents (Section 3.8). In accord with our previous simulations, the geodesic synergy hypothesis assigns a fixed processing time interval of 100 ms to the Riemannian OTG. In that time the OTG reads-in, from higher levels in a response-planning hierarchy, the initial position and velocity \( (s(t_i), \dot{s}(t_i)) \) and the final position and velocity \( (s(t_f), \dot{s}(t_f)) \) along the geodesic pathway for a BUMP, generates an optimal trajectory \( (s(t), \dot{s}(t)) \) with minimum metric-acceleration to connect \( (s(t_i), \dot{s}(t_i)) \) with \( (s(t_f), \dot{s}(t_f)) \) along the geodesic pathway, and holds it ready for execution. While one BUMP is being executed in real-time, the next is being generated in parallel. Because of distributed processing within the OTG the trajectory \( s(t), \dot{s}(t) \) can have a duration much longer than the 100 ms required by the OTG to generate it. Consequently, after only the first 100 ms of the trajectory has been executed the OTG has had sufficient time to generate a completely new required response trajectory with a new prediction horizon \( t_f - t_i \) if required. Thus movement along the geodesic pathway \( x(s) \) is comprised of a concatenated sequence of 100 ms duration, minimum metric-acceleration submovements with each submovement able to correct for execution errors detected during the previous 100–200 ms. Strategic choice of prediction horizon \( t_f - t_i \) its effect on the smoothness of the total concatenated trajectory, and other issues concerning error correction have all been examined in detail in previous work (Bye & Neilson, 2008, 2010; Neilson & Neilson, 2005a; Neilson et al., 1995; Oytam et al., 2005). Those matters are not affected by the substitution of metric equivalents for their Euclidean counterparts and so are not further addressed here.

6.3. Minimum energy temporal flow for a multiple-CDOF geodesic movement

For a task requiring movement with two or more CDOFs the geodesic synergy hypothesis holds that the temporal trajectories for each CDOF are generated independently and in parallel. This requires the parallel operation of OTG circuitry dedicated to each CDOF, a proposal that is a key part of the BUMP formulation (Section 6.1, point vii). As such, the rationale for this proposal is already available (see Bye & Neilson, 2008; Neilson & Neilson, 2005a; Oytam et al., 2005). As set out in Sections 5.3 and 5.4, a multiple-CDOF geodesic movement synergy is confined spatially to a totally geodesic submanifold of equivalent dimension embedded in the configuration manifold \( (C, J) \). Using geodesic coordinate grid lines, the initial and final positions and velocities in the submanifold can be projected on to the \( N \) geodesic coordinate axes that span the submanifold. \( N \) OTGs working independently and in parallel can then generate minimum metric-acceleration trajectories along each of the geodesic coordinate axes to connect the projected initial and final positions and velocities in the specified time \( t_f - t_i \). This is possible because temporal flow along a geodesic pathway can be scaled independently of the pathway. In other words, a minimum metric-acceleration trajectory is planned independently for each of the \( N \) CDOFs.

Essentially, for generating a minimum muscular effort temporal flow to connect an initial position and velocity \( (x(t_i), \dot{x}(t_i)) \) and a final position and velocity \( (x(t_f), \dot{x}(t_f)) \) within an \( N \)-dimensional submanifold, what we have is an \( N \)-fold version of the problem already solved for optimizing \( s(t) \) and \( \dot{s}(t) \) in the one-CDOF case. The required initial and final positions and velocities \( (x(t_i), \dot{x}(t_i)) \) and \( (x(t_f), \dot{x}(t_f)) \) can be projected on to each of the geodesic coordinate axes spanning the submanifold. With \( N \) OTGs working independently in parallel the constituent minimum metric-acceleration BUMPs for each CDOF along each of the geodesic coordinate axes can be generated independently and in parallel and held ready for execution. When the \( N \) BUMP trajectories are executed together by the multivariable feedforward–feedback movement control system described in AMT (for detail see Neilson & Neilson, 2005a) the resulting movement follows a minimum metric-acceleration trajectory confined within the \( N \)-dimensional totally geodesic submanifold. (A simplified illustration of this temporal planning process is given in Section 7.2.)

7. The geometry of minimum energy response planning

The concepts of minimum energy spatial response planning (Section 5) and minimum energy temporal response planning (Section 6) provide the basis for the geodesic synergy hypothesis. To obtain a picture of the processes involved in temporal response planning we now examine their application to the simplified tasks of generating geodesic trajectories and minimum metric-acceleration trajectories on a sphere. Note that the sphere is used here only to provide a simple example of temporal planning on a curved manifold. In no way is it meant to imply that the Riemannian metric on a sphere equals the
mass-inertia matrix of the human body or any part of the human body for that matter. Also note that we are using the term ‘sphere’ according to its mathematical definition as the two-dimensional surface of a three-dimensional ball. Moreover, the two-dimensional sphere is regarded as an abstract two-dimensional manifold in its own right (i.e., not as a submanifold embedded in a three-dimensional Euclidean space or indeed in any other higher dimensional manifold). This sphere can be endowed with a metric that allows the construction of geodesic pathways known as ‘great circles’. These include horizontal and vertical geodesic coordinate grid lines. The procedure for generating minimum metric-acceleration pathways to connect specified initial and final positions and velocities on the sphere is the same as for any two-dimensional totally geodesic submanifold on which horizontal and vertical geodesic coordinate grid lines can be constructed. Thus we can use the sphere with its great circle geodesics to provide a simple illustration of the processes involved in temporal response planning on any such totally geodesic submanifold.

7.1. Minimum energy spatial response planning on a sphere

A picture of the geometry involved in planning geodesic movement trajectories on curved totally geodesic submanifolds can be obtained by considering the analogous but simpler problem of finding the minimum distance pathway to navigate from point A to point B on a sphere. Planning a minimum kinetic energy pathway to connect configurations A and B on a sphere is equivalent to finding the minimum distance pathway on the sphere between the two points. Both involve finding the unique geodesic on the sphere that passes through A and B. It is well known that the geodesic (shortest distance) pathway between any two points on a sphere is equal to the ‘great circle’ route connecting the two points. Suppose points A and B have different longitudes but the same latitude, as in Fig. 2.

Lines of longitude are great circles on the sphere but lines of latitude (apart from the equator) are not. Moving along the latitude line connecting A and B is, therefore, not the shortest pathway between the two points. We can use the equator and one of the longitude lines as geodesic coordinate axes \( x(x^1) \) and \( \beta(x^2) \) respectively (using the same notation as used in Figs. A1, A2, and A3 in Appendix A), and then we can construct a system of geodesic (great circle) horizontal and vertical coordinate grid lines, as illustrated in Fig. 2. These geodesic coordinate grid lines are a property of the sphere and its metric and can be determined in advance. Using these great circle grid lines, project the points A and B on to the geodesic coordinate axes \( x(x^1) \) and \( \beta(x^2) \). Now plan two constant speed trajectories (i.e., constant metric-speed across the surface of the sphere) to move from \( x(x^1)_A \) to \( x(x^1)_B \) along \( x(x^1) \) and from \( x(x^2)_A \) to \( x(x^2)_B \) along \( \beta(x^2) \) in the same interval of time \( \Delta t \). In other words, the constant metric-speeds along \( x(x^1) \) and \( \beta(x^2) \) are set equal to \( \frac{(x(x^1)_B - x(x^1)_A)}{\Delta t} \) and \( \frac{(x(x^2)_B - x(x^2)_A)}{\Delta t} \), respectively. When these two constant speed trajectories are projected back along the great circle grid lines they form the required constant metric-speed great circle (geodesic) trajectory connecting A to B (Fig. 2). This procedure can be used to compute geodesic pathways from any point A to any point B on the sphere or any other totally geodesic submanifold on which horizontal and vertical geodesic coordinate grid lines can be constructed.

7.2. Minimum energy temporal response planning on a sphere

Most functional movements require accelerations and/or decelerations. Moving a finger from A to B, for example, requires the limb first to accelerate and then to decelerate. In making such a movement it is possible to utilize the accelerations and decelerations caused by gravity, support forces and the changing mass-inertia characteristics of the body to minimize demand for muscular effort. An intuitive understanding of the geometry involved in planning a minimum energy temporal trajectory between a specified initial position and velocity and a specified final position and velocity for each submovement (BUMP) in a totally geodesic submanifold (selected by spatial planning) can be obtained by extending the sphere analogy from the previous section. We wish to find a minimum metric-acceleration trajectory to move between an initial point A and initial velocity \( v_A = 0 \) on the sphere illustrated in Fig. 3) to a final point B and final velocity \( v_B \) on the sphere.

Project these initial and final positions and velocities on to the geodesic coordinate axes \( x(x^1) \) and \( \beta(x^2) \) using the great circle geodesic coordinate grid lines as in Fig. 2 giving \( (x(x^1)_A, x(x^1)_B) \) and \( (x(x^2)_A, x(x^2)_B) \) on \( x(x^1) \), and \( (\beta(x^2)_A, \beta(x^2)_B) \) and \( (\beta(x^2)_A, \beta(x^2)_B) \) on \( \beta(x^2) \) (Fig. 3). Now plan a minimum metric-acceleration trajectory to move between \( (x(x^1)_A, x(x^1)_B) \) and \( (x(x^1)_B, x(x^1)_B) \) on \( x(x^1) \) in a specified interval of time \( \Delta \tau \), and a minimum metric-acceleration trajectory to move between \( (\beta(x^2)_A, \beta(x^2)_A) \) and \( (\beta(x^2)_B, \beta(x^2)_B) \) on \( \beta(x^2) \) in the same interval of time \( \Delta \tau \). Providing metric-distances, metric-velocities, and metric-accelerations are employed, these are exactly the trajectories generated by the optimum trajectory generator (OTG) described previously in the BUMP model of response planning (Bye & Neilson, 2008, 2010; Neilson et al., 1995). Two OTGs working independently and in parallel generate the two trajectories along coordinate axes \( x(x^1) \) and \( \beta(x^2) \) in no more time than required to generate one trajectory. When the two trajectories are projected back along the great circle grid lines they form the required minimum metric-acceleration trajectory connecting the initial point A and initial velocity \( v_A \) (zero in Fig. 3) to the final point B and final velocity \( v_B \) on the sphere, as illustrated in Fig. 3. To meet the endpoint velocity requirements the two component trajectories have to accelerate and/or decelerate so they are no longer constant metric-velocity trajectories and therefore are no longer geodesic trajectories. Likewise the resulting trajectory is no longer a great circle geodesic pathway but nevertheless, it is confined to the surface of the sphere and it is the smooth trajectory connecting specified initial and final positions and velocities on the sphere that minimizes metric-acceleration. Remember, the minimum metric-acceleration temporal planning procedure illustrated here on a sphere applies to each BUMP in a smoothly
connected concatenated sequence of BUMPs. The position and velocity at the end of one BUMP is set equal to the position and velocity at the beginning of the next and correction of execution errors detected by sensory feedback during the execution of one BUMP can be incorporated into the planning of a subsequent BUMP.

Fig. 2. Generating a great circle geodesic pathway to connect arbitrary points A and B on a sphere. The equator and a longitude line form geodesic coordinates \( a(x^1) \) and \( b(x^2) \), respectively. The coordinate grid lines are all predetermined geodesic pathways on the sphere. \( x^1 \) is metric-distance along \( a(x^1) \) and \( x^2 \) is metric-distance along \( b(x^2) \). Point A is projected along geodesic coordinate grid lines to point \( (x^1)_A \) on coordinate axis \( a(x^1) \) and to point \( (x^2)_A \) on coordinate axis \( b(x^2) \). Similarly, point B is projected along geodesic coordinate grid lines to point \( (x^1)_B \) on \( a(x^1) \) and \( (x^2)_B \) on \( b(x^2) \). Constant metric-speed trajectories are generated to move from \( (x^1)_A \) to \( (x^1)_B \) along \( a(x^1) \) in a specified interval of time \( D_t \) and from \( (x^2)_A \) to \( (x^2)_B \) along \( b(x^2) \) in the same interval of time \( D_t \). When these two trajectories are projected back along geodesic coordinate grid lines they generate the unique geodesic pathway on the sphere connecting point A to point B. See text for details.

Fig. 3. Planning a minimum metric-acceleration trajectory to connect an initial point A with initial velocity \( v_A = 0 \) to a final point B with final velocity \( v_B \) on a sphere. The geodesic coordinate axes and geodesic coordinate grid lines are the same as in Fig. 5. Point A is projected along geodesic coordinate grid lines to the point \( (x^1)_A \) on \( a(x^1) \) and \( (x^2)_A \) on \( b(x^2) \). Point B and final velocity \( v_B \) are projected along geodesic coordinate grid lines to position and velocity \( (x^1)_B, (x^2)_B \) on \( a(x^1) \) and to position and velocity \( (x^1)_B, (x^2)_B \) on \( b(x^2) \). Two optimum trajectory generators (OTGs) working independently and in parallel generate minimum metric-acceleration trajectories to connect \( (x^1)_A \) to \( (x^1)_B \), \( (x^2)_A \) along \( a(x^1) \) and \( (x^2)_A \) to \( (x^2)_B \) along \( b(x^2) \) in the same interval of time \( D_t \). To achieve the final velocities \( (x^1)_B \) and \( (x^2)_B \) the trajectories have to overshoot the positions \( (x^1)_A \) and \( (x^2)_A \), slow down, reverse, and approach the final positions \( (x^1)_A \) and \( (x^2)_A \) from the opposite directions. When these trajectories are projected back along geodesic coordinate grid lines they generate the minimum metric-acceleration trajectory on the sphere connecting point A with initial velocity \( v_A = 0 \) to point B with final velocity \( v_B \). See text for details.
As mentioned above and in Section 6, this temporal planning procedure generalizes to planning of minimum metric-acceleration trajectories on any totally geodesic $N$-dimensional submanifold embedded in the curved Riemannian configuration manifold $(C, J)$ on which geodesic coordinate grid lines can be constructed.

8. GTG simulator

The geodesic synergy hypothesis is illustrated using a MATLAB/Simulink simulation of the GTG to generate two-dimensional totally geodesic coordinate systems for a two-DOF arm moving in the horizontal plane. We use the same two-DOF arm described by Biess et al. (2011), the details of which are as follows:

The two-dimensional configuration manifold $(C, J)$ for the arm is spanned by the shoulder angle $\theta_1$ and the elbow angle $\theta_2$. The velocity vector $v$ at configuration $c = (\theta_1, \theta_2)$ is $v = (\dot{\theta}_1, \dot{\theta}_2)$. The mass-inertia matrix (i.e., kinetic-energy metric) is:

$$J(\theta_2) = \begin{bmatrix} I_{11}(\theta_2) & I_{12}(\theta_2) \\ I_{12}(\theta_2) & I_{22}(\theta_2) \end{bmatrix} = \begin{bmatrix} I_1 + I_3 + 2I_5 \cos \theta_2 & I_3 + I_5 \cos \theta_2 \\ I_3 + I_5 \cos \theta_2 & I_3 \end{bmatrix}.$$  

(7)

where constants are: $I_1 = I_{1x} = m_1 l_1^2 + m_2 l_2^2; \quad I_3 = I_{2x} + m_2 l_2^2; \quad I_5 = m_2 l_2 a_2; \quad I_{1x} = \text{moment of inertia of upper arm}; \quad I_{2x} = \text{moment of inertia of forearm}; \quad l_1 = \text{length of upper arm}; \quad l_2 = \text{length of forearm}; \quad m_1 = \text{mass of upper arm}; \quad m_2 = \text{mass of forearm}; \quad a_1 = \text{distance to centre of mass of upper arm}; \quad a_2 = \text{distance to centre of mass of forearm}. In the simulation these parameters are set to the same values used by Biess et al.: $l_1 = 0.30 m, l_2 = 0.345 m, m_1 = 2.52 kg, m_2 = 2.07 kg, a_1 = 0.142 m, a_2 = 0.225 m, I_{1x} = 0.019 kg m^2, and I_{2x} = 0.021 kg m^2.$ The inertial constants in $\text{Eq. (7)}$ are derived assuming the arm to be rigidly supported at the shoulder girdle. The mass-inertia matrix $J(c)$ changes as a function of elbow angle $\theta_2$ and consequently the configuration manifold $(C, J) = ((\theta_1, \theta_2), J(\theta_2))$ is considered a curved Riemannian manifold with $J(\theta_2)$ equal to the kinetic-energy metric tensor. Taking the limited range of joint-angle change into account, the configuration manifold $(C, J)$ corresponds to a path on the surface of a two-torus.

The GTG simulator (Fig. 5) generates constant metric-speed geodesic trajectories in the curved Riemannian configuration manifold $(C, J) = ((\theta_1, \theta_2), J(\theta_2))$ starting from any initial position $c(0) = (\theta_1(0), \theta_2(0))$ and any initial velocity $\dot{c}(0) = (\dot{\theta}_1(0), \dot{\theta}_2(0))$ at $t = 0$. These are set as initial values $\theta_1(0), \theta_2(0), \dot{\theta}_1(0), \dot{\theta}_2(0)$ on the integrators in Fig. 5. (Note that in the MATLAB/Simulink notation of the figure these are written as theta1_init, theta2_init, theta1dot_init, and theta2dot_init respectively.) The outputs $\theta_1(t), \theta_2(t), \dot{\theta}_1(t),$ and $\dot{\theta}_2(t)$ of the integrators are sampled at $T = 50$ ms sample intervals (i.e., 20/s) by zero-order hold (ZOH) samplers as shown in Fig. 5 and the sampled signals $\dot{\theta}_1(k), \dot{\theta}_2(k), \dot{\theta}_1(k),$ and $\dot{\theta}_2(k),$ where $k = \text{sample number}$, are applied to the input of an association memory network labelled $\text{TTC}$ in Fig. 5 where $\text{TTC}$ stands for double tangent bundle of the configuration manifold. The zero-order hold samplers are included in the simulator to approximate a 20 Hz bursting code of cortical columns in the brain but this bursting frequency can be easily changed. Pre-computed values $[f_1, f_2, f_3]$ of the acceleration geodesic spray vector $f_2(c, v)$ are retrieved from the association memory $\text{TTC}$ and applied to the inputs of the double integrators, as shown in Fig. 5, where they are held constant for one sample interval $T = 50$ ms

until the next sample is taken by the ZOHs. The acceleration geodesic spray vector $f_2(c, v) = [f_1, f_2, f_3]^{T}$ equals the natural free-motion acceleration of the arm in the absence of an external force. It is caused by conservation of kinetic energy and the changing mass-inertia $J(\theta_2)$ of the arm. It can be pre-computed for each configuration $c = (\theta_1, \theta_2)$ and velocity $\dot{c} = v = (\dot{\theta}_1, \dot{\theta}_2)$ as shown below and stored in $\text{TTC}$. The kinetic energy $K$ is given by the metric inner product $K = \frac{1}{2} J(c)|v|^2$. Substituting for $K$ into Eq. (3) and rearranging gives

$$\langle f(c)v, w \rangle = \frac{1}{2} \langle f(c)w, v \rangle - \langle f(c)v, v \rangle,$$

(8)

where $w$ is an arbitrary fixed vector. Solving Eq. (8) for the unknown $f_2(c, v)$ with $w = [1, 0]^T$ and again with $w = [0, 1]^T$ gives

$$f_2(\theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{\text{det}[J]}{J_{12}J_{32} - J_{13}J_{22}} m_2 l_1 a_2 \sin(\theta_2) \left( \frac{\dot{\theta}_1}{\dot{\theta}_2} \right)^2 - \frac{\text{det}[J]}{J_{11}J_{32} - J_{12}J_{31}} 2m_2 l_1 a_2 \sin(\theta_2) \frac{\dot{\theta}_1}{\dot{\theta}_2}.$$ 

(9)

where $\text{det}[J] = J_{11}J_{32} - J_{13}J_{22}$.

The GTG simulator generates constant metric-speed geodesic trajectories in the curved Riemannian configuration manifold $(C, J) = ((\theta_1, \theta_2), J(\theta_2))$ and these appear as accelerating or decelerating curves in joint-angle space. But we are also interested in the corresponding geodesic trajectories of hand position in the $(x, y)$-plane. Therefore we have included in the simulator the following two-way transformations between the $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ and the $(x, y, \dot{x}, \dot{y})$ coordinates:

$$\begin{align*}
x &= l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\
y &= l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2)
\end{align*}$$

(10)
\[
\begin{align*}
\frac{x}{y} &= \begin{bmatrix}
-l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) \\
l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix} \\
\theta_1 &= \text{atan2}(y, x) - \cos \left(\frac{d^2 + l_1^2 - l_2^2}{2l_1d}\right) \\
\theta_2 &= \pi - \cos \left(\frac{l_1^2 + l_2^2 - d^2}{2l_1l_2}\right)
\end{align*}
\]

where \( d = (x^2 + y^2)^{\frac{1}{2}} \).

\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix} = \begin{bmatrix}
-l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) \\
l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)
\end{bmatrix}^{-1} \begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix}
\]
Using these coordinate transformations the simulator generates corresponding geodesic trajectories in both the two-dimensional joint-angle space \((\theta_1 - \theta_2)\) and the two-dimensional \((x-y)\)-plane of hand position, as in Fig. 6.

Fig. 6(a) shows a two-dimensional totally geodesic coordinate system spanning the two-dimensional task space for the hand moving in the horizontal \((x-y)\)-plane. The coordinate system is centred about the point \((x, y) = (0\text{ m}, 0.33\text{ m})\) with initial velocities set to \((\dot{x}, \dot{y}) = (0.2\text{ m/s}, 0\text{ m/s})\) for the horizontal \((x)\) geodesic coordinate axis and \((\dot{x}, \dot{y}) = (0\text{ m/s}, 0.2\text{ m/s})\) for the vertical \((\beta)\) geodesic coordinate axis. The \(x\) and \(\beta\) geodesic coordinate axes (i.e., geodesic pathways) in the \((x-y)\)-plane are similar to those described by Biess et al. (2011). The initial positions and velocities for the horizontal and vertical geodesic coordinate grid lines are obtained by parallel translating the initial positions and velocities along the \(\beta\) geodesic coordinate axis and the \(x\) geodesic coordinate axis, respectively. Fig. 6(b) shows the corresponding geodesic coordinate axes and geodesic coordinate grid lines plotted in joint-angle space \((\theta_1 - \theta_2)\). Movement of the hand to the right in the \((x-y)\)-plane corresponds to movement to the left in joint-angle space and upward movement of the hand in the \((x-y)\)-plane corresponds to downward movement in joint-angle space. This is because of the way the shoulder and elbow angles are defined (see Fig. 4). While the geodesics follow curved pathways in joint-angle space the corresponding geodesic hand paths are more or less straight in the \((x-y)\)-plane. The totally geodesic coordinate system corresponds to a two-DOF minimum-effort geodometric movement synergy that can be used for planning minimum-effort movements of the hand in two-dimensional tasks such as two-dimensional tracking and handwriting, as well as for point-to-point reaching movements like those described by Biess et al. (2011).

Fig. 6(c) and (d) mirrors Fig. 6(a) and (b). Fig. 6(c) shows a two-dimensional totally geodesic coordinate system spanning the two-dimensional joint-angle space. The coordinate system is centred about the point \((\theta_1, \theta_2) = (0.4152\text{ rad}, 2.0757\text{ rad})\) in joint-angle space with initial velocities set to \((\dot{\theta}_1, \dot{\theta}_2) = (0.5\text{ rad/s}, 0\text{ rad/s})\) for the horizontal \((\alpha)\) geodesic coordinate axis and \((\dot{\theta}_1, \dot{\theta}_2) = (0\text{ rad/s}, 0.5\text{ rad/s})\) for the vertical \((\beta)\) geodesic coordinate axis. The initial positions and velocities for the horizontal and vertical geodesic coordinate grid lines are obtained by parallel translating the initial positions and velocities along the \(\beta\) geodesic coordinate axis and the \(x\) geodesic coordinate axis, respectively, in joint-angle space. Fig. 6(d) shows the corresponding geodesic coordinate axes and geodesic coordinate grid lines plotted in the two-dimensional \((x-y)\)-space for hand movement. While the geodesic coordinates in Fig. 6(d) are skewed relative to those in Fig. 6(a) because of the different initial positions and velocities used to generate them they are still more or less straight geodesic hand paths and both geodesic coordinate systems span the same two-dimensional \((x-y)\)-space. The relative spacing of the geodesic coordinate grid lines for geodesic hand paths in Fig. 6(a) and (d) shows that some warping of the \((x-y)\)-space for geodesic hand paths occurs, particularly towards the boundary of the task space. This warping is caused by the changing mass-inertia metric \(J(\theta_2)\) and the resulting nonlinear inertial, centrifugal and Coriolis reaction forces at work in the two-DOF arm. All geodesic pathways in joint-angle space are curved and the relative spacing between the geodesic coordinate grid lines provides a measure of the warping of joint-angle space caused by the kinetic-energy metric \(J(\theta_2)\). The joint-angle geodesic coordinates in Fig. 6(b) are skewed relative to those in Fig. 6(c) because of the different initial joint-angle positions and velocities used to generate them. Nevertheless, both geodesic coordinate systems span the same joint-angle task space, just as the hand-path geodesics in Fig. 6(a) and (d) span the same two-dimensional \((x-y)\)-plane defining the task space for hand position.

9. Discussion

9.1. Moving in the absence of muscle force?

Anyone who recalls a certain old give-away toy in the cereal pack will know it is possible to build simple mechanical devices that can walk without force generators (other than gravity) or movement control systems. Yet it would be absurd to conclude from this that the body does not need muscles or a movement control system. Nevertheless, we have seen that in the absence of all external forces, including visco-elastic and muscle forces, the mass-inertia of the body causes it to follow a free-motion trajectory that conserves kinetic energy. Indeed, given an appropriate initial velocity, it is possible theoretically to ‘glide’ from any configuration of the body to any other configuration without needing muscle force (i.e., assuming visco-elastic forces, gravitational forces, support forces, and all other external forces are zero). It seems reasonable, therefore, that free-motion trajectories determined by the mass-inertia properties of the body provide a road map for the most efficient pathways for moving about in configuration space with minimal demand for muscular effort.

In this paper we have shown that Riemannian geometry provides a suitable mathematical framework for computing minimum effort (free-motion) trajectories to move about in a local environment taking the influence of gravity and mechanical interactions with objects in the environment into account. Movement is conceptualized as a trajectory in a 116-dimensional curved Riemannian configuration manifold representing body posture plus the position and orientation (rotation) of the head in the environment. The configuration manifold is endowed with two predetermined vector (tensor) fields: (i) A Riemannian-metric tensor field or matrix field equal to the mass-inertia matrix of the body in each configuration. (ii) An acceleration vector field known as the geodesic spray at each position and velocity on the tangent bundle. The curvature of the configuration manifold at each point is defined by the first and second derivatives of the metric tensor (i.e., mass-inertia matrix) at each point. Given the Riemannian configuration manifold with its geodesic spray field, and using methods of Riemannian geometry, we have shown how to compute multi-joint coordinations (movement synergies) and minimum metric-acceleration movement trajectories within those movement synergies to achieve movement goals with minimum
demand for muscular effort. Over time, through trial and error, imitation, and coaching, people find the most energy efficient way of achieving task goals; for example, walking up stairs or reaching across a table. The coordination (or movement synergy) that achieves the task goal(s) with minimum muscular effort corresponds to a submanifold in the configuration manifold of the body. It is a special submanifold known as a ‘totally geodesic submanifold’. It is possible, however, (but by no means proven given that a person can search all body configurations to find the most energy efficient way of performing a task), that a totally geodesic submanifold compatible with a nominated task space might not exist. In this case the most energy efficient movement synergy compatible with the task space corresponds to the submanifold that deviates minimally from a totally geodesic one.

9.2. Controlled falling

9.2.1. A generalized equilibrium hypothesis

With initial position and velocity specified, double integration of the predetermined geodesic spray acceleration field gives a curve in the configuration manifold corresponding to the natural free motion of the body. This curve is known as a geodesic of the Riemannian connection. We saw in Section 5.5 that a geodesic pathway connecting specified initial and final points in the manifold corresponds to the minimal muscular effort pathway connecting the two points. However, for movements within a gravitational field, muscles have to generate changing force at every point along the curve to cancel the changing gravity and the support forces acting on the body. Only under these conditions when the net external force is zero does the free motion of the body follow a geodesic trajectory. This leads to the contention that movement is planned as the flow of an equilibrium point along geodesic pathways in the configuration manifold. This can be seen as a generalization of Feldman’s equilibrium point hypothesis (Feldman, 1986, 2009; Feldman & Levin, 1995; Feldman, Ostry, Levin, Gribble, & Mitnitski, 1998).

**Fig. 6.** Plots of geodesic coordinate axes and geodesic coordinate grid lines in the (x-y)-plane and joint-angle space for two-CDOF geodesic movement synergies generated by the GTG simulator. Spacing of geodesic grid lines in all plots equals distance travelled in 100 ms. Details are given in the text. The two-dimensional totally geodesic coordinate systems can be used for planning minimum-effort movements of the arm and hand anywhere in the two-dimensional task space of the two-DOF arm. (a) shows a totally geodesic coordinate system for the hand in the (x-y)-plane centred at (x, y) = (0 m, 0.33 m) with initial velocity for the horizontal (alpha) geodesic coordinate axis set to (x, y) = (0.2 m/s, 0 m/s) and initial velocity for the vertical (beta) geodesic coordinate axis set to (x, y) = (0 m/s, 0.2 m/s), that is, the initial velocities are orthogonal in the (x-y)-plane. (b) shows the corresponding geodesic coordinate axes and coordinate grid lines in joint-angle space. (c) shows a totally geodesic coordinate system for the arm in joint-angle space centred at the point (h1, h2) = (0.4152 rad, 2.0757 rad) with initial velocities for the horizontal (alpha) geodesic coordinate axis set to (h1, h2) = (0.4152 rad, 2.0757 rad) and initial velocity for the vertical (beta) geodesic coordinate axis set to (h1, h2) = (0.5 rad/s, 0 rad/s), that is, initial velocities are orthogonal in joint-angle space. (d) shows the corresponding geodesic coordinate axes and geodesic coordinate grid lines in the (x-y)-plane.
A generalized equilibrium point hypothesis greatly simplifies generation of movement trajectories and allows them to be
time-scaled, but it implies that the body can be held in a state of equilibrium at every point along every geodesic curve in the
manifold. This is unrealistic! Many configurations of the body cannot be maintained in a state of equilibrium and many pur-
purpose movements deliberately include non-equilibrium configurations. For example, during walking the centre of pressure
is deliberately moved outside the support base allowing the body to fall forward and hence deviate away from equilibrium.
The fact that not all points in the configuration manifold are equilibrium points presents a challenge to a generalized equi-
librium point hypothesis that requires further consideration.

9.2.2. Impossible postures and no-go places

In usual circumstances the nervous system does not plan movements that might damage joints and ligaments by
attempting to move at high speed beyond the anatomical limits of the body’s range of movement. This range of movement
varies with configuration in a complicated way because in many configurations it is limited not by the range of each elemen-
tal movement per se, but by one part of the body colliding with another. Similarly, possible configurations of the body are
limited by collisions with objects in the environment and by ‘no-go’ places in the environment. It is not possible, for example,
to walk through a brick wall, or to float up into the air. Such impossible postures and no-go places impose a boundary,
∂(C, J), on the configuration manifold (C, J). A 116-dimensional manifold with boundary can be thought of as the union
of two disjoint (nonintersecting) submanifolds (Lee, 2013); a 116-dimensional interior submanifold Int(C, J) and a 115-
dimensional boundary submanifold, ∂(C, J).

To prevent falling, the body has to be supported against gravity. This requires mechanical interaction between the body
and support surfaces in the environment. In other words, most but not all functional movements involve mechanical inter-
actions with objects in the environment and are contained, therefore, within the 115-dimensional boundary submanifold
∂(C, J). Configurations within the interior submanifold Int(C, J) are unsupported and consequently, the body must be falling
and cannot be in a state of equilibrium. It follows that for an equilibrium point to flow along a geodesic pathway, the path-
way must be contained within the boundary ∂(C, J) of the manifold.

To elaborate the above distinction, collisions between parts of the body and with objects in the environment (e.g., support
surfaces) impose a boundary in the configuration manifold that separates it into configurations that can be reached and
those that cannot. Whenever the body is prevented from falling by a support surface (e.g., standing on the floor) it has to be
in contact with that surface. Therefore the configuration of the body has to be contained within the 115-dimensional
boundary submanifold ∂(C, J) and not in the 116-dimensional interior submanifold Int(C, J). For analogy, think of the
two-dimensional boundary (i.e., a sphere) of a closed ball in three-dimensional space. The ball has an open three-
dimensional interior. While many different configurations can be assumed by the body while being supported by the surface
it is not possible for the body to move through the support surface; this reduces the dimension of the boundary submanifold
by one to 115. Most (but not all) movements take place with the body supported against gravity and so involve configura-
tions of the body contained within the 115-dimensional boundary submanifold. For example, consider a pointing movement
executed by a four-DOF model of the arm similar to that described by Biess et al. (2011). The arm is assumed to be rigidly
supported at the shoulder with no movement of the clavicle or scapula. Actually, stabilization of the shoulder and the rest of
the body requires synergistic activation of many muscles throughout the body in a chain reaction until reaction forces and
torques are encountered between the body and a support surface. This nicely illustrates why whole-body movement syn-
dnergies have to be taken into account. During the pointing movement of the arm the body is supported against gravity by
the support surface and so the configurations of the body associated with the pointing movement are contained within
the 115-dimensional boundary submanifold ∂(C, J).

9.2.3. Purposive falling

The contention that movements are planned as the flow of an equilibrium point along a geodesic pathway simplifies the
task of generating minimum-effort movement trajectories. However, as mentioned above, many functional movements (e.g.,
walking, running, and jumping that include falling as part of the movement) pass through configurations that are not equi-
librium points. Indeed, whenever the body is in the interior Int(C, J) of the configuration manifold it is not in contact with a
support surface and so cannot be in equilibrium. This challenges the hypothesis because all inertial free-motion pathways (i.
e., geodesic pathways) have to be confined to configurations where the external forces acting on the body sum to zero. A non-
zero net force pushes the body away from equilibrium and hence away from geodesic pathways. Does this mean that the
geodesic synergy hypothesis cannot be applied to purposive movements that include falling under the action of an external
force field? This seems unlikely since deliberately allowing the body to fall as part of a purposive movement is a strategy for
reducing muscular effort and that, after all, is the objective of the hypothesis. How then can it be reconciled with the exis-
tence of purposive movements that deliberately include falling?

Movements of an astronaut in a state of weightlessness in an orbiting space station provide a possible answer. The space
station is in free fall. The astronaut senses his/her place and orientation relative to an external reference frame attached to
the space station and plans movements within that space, thereby cancelling the free-fall motion of the space station.

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8 A boundary on the place map $\mathcal{P}$ can lead to confusion as to whether the place map is three-dimensional or two-dimensional because the boundary usually
constrains vertical movement more than horizontal movement.
A similar thing happens when moving about within a moving vehicle such as a plane, train, or bus. The person senses place and orientation relative to the vehicle. The outside world is seen as moving past the window. People are able to reset the location of their external reference frame to suit the conditions and the task at hand. This is consistent with the resetting of grid cell coordinate maps observed in rats as they move from one compartment to another in a multi-compartmental environment (Derdikman & Moser, 2010; Derdikman et al., 2009). A high diver, for example, can employ geodesic synergies to plan and execute minimum effort twisting and somersaulting movements during the dive by utilizing an external reference frame that is centred on his/her body. In general, deviations from equilibrium associated with being transported about by a vehicle (e.g., a ballerina being carried by her partner) or by including falling movements in responses can be handled within the geodesic synergy hypothesis if the location of the origin of the external reference frame can be appropriately reset.

This raises the question of how the nervous system senses its place and orientation in the environment and what causes orientation maps to reset from one environmental compartment to another? This is an important area of research (Fyhn, Molden, Witter, Moser, & Moser, 2004; Hafting, Fyhn, Molden, Moser, & Moser, 2005; O’Keefe & Dostrovsky, 1971; O’Keefe, 1976; Sargolini et al., 2006) and much is already known about the neuroscience of this spatial sense. It does not seem unreasonable that in a gravitational field a sense of vertical and horizontal can be obtained from the labyrinthine system. However, this leaves the sense of direction (compass) unexplained and it does not account for the resetting of spatial maps from one local environment to another. Positions of objects in the local environment as perceived visually and the way objects appear to change position relative to each other with movement of the head in the environment are likely to play a role. See Section 9.5 for discussion of this.

9.3. Comparisons with Riemannian geometry model of Biess

We now contrast our approach with that of Biess (Biess et al., 2011; Biess, 2013) having shown in Section 8 a correspondence between results for a two-DOF arm movement. Biess et al. (2011) solved a two-point boundary value problem to generate the unique geodesic trajectory for the kinetic-energy metric to move between two points in the four-DOF arm configuration space with minimum muscular effort. Subsequently, Biess (2013) has considered arm movements in which the hand is constrained to move between two points on a surface (e.g., a sphere) or along a curve (e.g., an ellipse). Such constrained arm movements no longer correspond to geodesic trajectories for the kinetic-energy metric for the arm. Biess solves this problem by computing a new metric for the arm configuration space that incorporates the metric of the particular constraining surface or curve. He refers to this as shaping the arm configuration space by prescription of non-Euclidean metrics. He then computes the geodesic trajectory for the new metric to move between the specified initial and final points. This solution requires a different metric to be calculated for each type of constrained arm movement and the resulting movement no longer corresponds to a minimum muscular effort trajectory. The constrained arm movement problem is solved, therefore, by a one-parameter geodesic movement trajectory but with multiple arm configuration space metrics, one for each type of movement. Biess (2013) went on to show that his metrically adjusted configuration space of the arm produces constrained movement that conforms to the one-third power law. Supported by much experimental evidence, this holds that the tangential speed-curvature relationship.

Again, the selection of the appropriate movement synergy imposes the necessary constraints while keeping the same minimum muscular effort kinetic-energy metric. We have shown previously (Neilson et al., 1995) that the method for planning and executing a sequence of constrained tracing submovements within a multi-DOF movement synergy likewise reproduces the speed-curvature relationship. Using the example of tracing a handwritten letter ‘b’ we modelled a smoothly connected sequence of short duration submovements each planned with minimum acceleration and executed on a two-dimensional surface. We compared the simulated movement trajectories with actual tracing movements of human subjects tracing the handwritten letter ‘b’ measured using an electronic pen on a bit-pad (digitizing tablet). The match between the simulated movement trajectories and the actual trajectories was so good they could be superimposed one on top of the other. The speed-curvature profiles matched even around the sharpest corner in the handwritten character ‘b’. Applying a bound on the accuracy of the tracing for each submovement not only determined the average tangential speed of the tracing (the slower the average tangential speed the more accurate the tracing), but scaled the relationship between the tangential speed and the curvature to ‘slow down around sharp corners’. This earlier work effectively demonstrates that a sequence of minimum effort submovements constrained within a two-DOF movement synergy accords very satisfactorily with the speed-curvature relationship.
In the context of whole-body movement to which the geodesic synergy hypothesis applies, the constraint for the hand to move on a particular surface or to trace a particular curve is no different from the constraint of remaining seated while executing a task or of not colliding with objects in the environment. The hypothesis effectively absorbs the two-point boundary problem and the problem of constraint into the selection of the geodesic synergy appropriate for the task space. Because this preserves the kinetic-energy metric throughout, movement proceeds with minimum muscular effort for whatever task is required, plus it is executed as a sequence of submovements which is characteristic of human behaviour in other than specifically ballistic tasks.

9.4. Neural plausibility

We have sought to present a conceptualization of synergistic movement of the entire human body that takes nonlinear dynamics into account and brings with it a viable set of computational procedures to effect that movement. Inevitably it must be asked: is this what the brain does and if so how? In the spirit of David Marr’s (Marr, 1982/2010) ‘three levels of understanding information processing systems’ we have an abstract computational theory of what the brain must do if it is to generate whole-body minimum effort coordinated movements taking nonlinear dynamics and interactions with the environment into account. What we do not have is a description of how the computations might be realized within the brain. Our suggestion is that smooth Riemannian manifolds (in particular the configuration manifold \((C, J)\)) can be seen as a geometric representation of the neural cognitive memory networks distributed throughout the neocortex and subcortical structures as described by Fuster (2008). We suggest that much of the flexibility and speed of the nervous system in planning and initiating coordinated, goal-oriented, multi-joint movements derives from its ability to store large amounts of information in distributed cognitive memory networks and to retrieve and reconstruct information quickly using parallel processing in these distributed networks.

A point \(c \in (C, J)\) in the configuration manifold \((C, J)\) corresponds to the posture of the whole body as well as the place and orientation of the body in the local environment. This is represented in the brain by a temporospatial pattern of neural activity distributed through parts of the somatosensory cortex, hippocampus and parahippocampal regions of the brain. This temporospatial pattern of neural activity, we suggest, is associated via synaptic modification over time with other temporospatial patterns of neural activity in other cortical networks distributed through other regions of the brain. The cortical association memory networks allow the distributed temporospatial pattern of neural activity encoding the configuration of the body to act as type of memory access code that points to and retrieves associated temporospatial patterns of neural activity in other cortical networks in the brain, just as a library catalogue points to the accession number that retrieves the right book. This picture of distributed cortical association memory networks is consistent with the place map and orientation map in the hippocampus and parahippocampus playing a role in the storage and retrieval of memories at sites distributed throughout the neocortex (O’Keefe, 2007; Stark, 2007). It is also consistent with Fuster’s (2008) description of cognitive cortical memory networks as being interactive and widely distributed throughout the cortex and subcortex with memory networks being formed by associative synaptic modulation through life experience.

In summary, our suggestion is that temporospatial patterns of neural activity retrieved from memory associated with neural activity representing body-configuration correspond to an encoding of the vectors and tensors at each \(c \in (C, J)\). These include the Riemannian-metric tensor field \(f(c)\) and the geodesic acceleration spray field \(f_2(c, v)\). It does not seem unreasonable to suggest that the nervous system can learn the mass and moment of inertia about elemental movements for various configurations of the body experienced through life and store this information in distributed cortical cognitive memory networks that update as the body changes via growth, decline, etc. Computing the geodesic spray field \(f_2(c, v)\) from Eq. (3) appears to require solving a set of 116 simultaneous, nonlinear, second-order differential equations. This might seem a daunting task and one may wonder whether it is possible to find a solution at all. However, when the computation is broken down into fibre-preserving calculations at each point \((c, v)\) on the tangent bundle \(TC\) it reduces to a relatively simple set of matrix operations (see Lang, 1999) and see computation of \((f_2^1 \ f_2^2)\) in Section 8. In other words, in Riemannian geometry, the distributed nature of the computation of \(f_2(c, v)\) greatly simplifies it but at a cost of having to perform a large number of simple calculations at each point on the manifold. Thus we see Riemannian geometry and the configuration manifold \((C, J)\) as providing a geometrical framework in which computational models of the neural functions attributed by Fuster (2008) to cortical cognitive networks can be formulated. Executive functions of those cognitive networks include working memory, preparatory set, and planning. These are exactly the functions described here in terms of Riemannian geometry and the configuration manifold \((C, J)\).

9.5. Visuospatial maps and synergy selection

For the nervous system to implement the Riemannian geometry model proposed by Biess (2013) to account for arm movements constrained to surfaces or curves, a new metric has to be computed for each constraining surface or curve and a set of eight nonlinear ordinary differential equations has to be solved for each point-to-point movement. Such equations require estimates of lengths, masses, centres of mass and moments of inertia of the upper arm and forearm and these are parameters not sensed directly by the nervous system. It seems unlikely the nervous system actually performs such a complicated computation to plan each pointing movement, especially when the brevity of the reaction time interval...
required to plan and initiate such a point-to-point movement is taken into consideration. Biess (2013) suggested that visual or proprioceptive input streams may play a role since their effect is to straighten hand paths, but how the nervous system might do this is not discussed. Indeed, he raises a number of questions: can sensorimotor representations be metric, does every motor task induce a different metric structure, and how can geometric information be extracted from sensorimotor input streams?

Our proposal is that through experience the nervous system builds up a repertoire of geodesic movement synergies and can switch quickly and smoothly from one previously learned movement synergy to another. If a task requires a movement synergy that has not been learned previously then initially the person is unable to perform the task with accuracy. Attempts at the movement are clumsy, slow and stiff. The person has to go through a learning process to acquire the new motor skill. Our formulation takes advantage of the fact that most of the computational workload involved in generating geodesic movement synergies can be carried out in advance and stored in distributed memory so only a small amount of computation (implemented by GTG and OTG trajectory generators) is required to plan and initiate each synergistic movement. This provides answers to Biess’s questions but raises new questions relating to sensory and cognitive processes that could realistically underpin the selection of a geodesic movement synergy compatible with task space. While we have made proposals about these processes previously (Neilson & Neilson, 2010), a detailed analysis of how the nervous system might achieve synergy selection in the context of nonlinear dynamics and Riemannian geometry requires consideration of sensory and cognitive processes beyond the present scope. Nevertheless, we can provide here a brief description of a fibre bundle conceptualization that provides a plausible means of synergy selection, to be elaborated fully in a subsequent paper.

Task space is usually specified in terms of visual and/or other sensory modalities. Selection of a geodesic synergy compatible with a task space requires, therefore, use of sensory-spatial maps. Let us consider visuospatial maps. Such maps can be represented in the Riemannian framework as a fibre bundle over the configuration manifold \((C, J)\). A fibre bundle is a straightforward extension of the configuration manifold \((C, J)\). At each point \(c \in (C, J)\) there exists a fibre containing, not a vector/tensor space, but a three-dimensional manifold \(G\) representing the visual space perceived when the body is in that configuration. The visual scene of the environment and of the body in that environment as seen from that configuration is represented by a vector field of visual features extracted by the visual system over the \(G\)-manifold. The fact that objects appear to shrink in size in proportion to depth is taken into account by endowing each \(G\)-manifold with a Riemannian metric equal to \(1/r^2I\), where \(r = \text{depth and } I\) is the identity matrix. Thus, at each \(c \in (C, J)\) there exists a memorized image of the environment and of the body in that environment as seen from that configuration, an image that is updated through experience. The street-view feature of Google maps provides an analogy. The point on the map resembles \(c \in (C, J)\) in the configuration manifold and all the street-view images at that point on the map resemble the visual field stored over the three-dimensional \(G\)-manifold at the point \(c \in (C, J)\). In this way the fibre bundle over \((C, J)\) forms a cognitive visuospatial map of the local environment and of the body moving in that environment. As above, the temporospatial pattern of neural activity encoding the configuration of the body acts as a memory access code that activates the appropriate visuospatial memory. Just as with Google street-view, feasibility of accomplishing this in the nervous system depends on the ability to store large amounts of readily accessible information in distributed memory networks.

The nervous system can build through experience memory associations between visual representations of task spaces and the symbols \((c, e_1, \ldots, e_N)\) representing the geodesic movement synergy appropriate for the task. Using the geometric methods described the geodesic coordinate system spanning the totally geodesic \(N\)-dimensional submanifold corresponding to the selected synergy \((c, e_1, \ldots, e_N)\) can be generated and stored. Then, via the fibre-bundle visuospatial map, the memorized visual images of the environment and of the body in that environment as seen from every configuration \(c\) within the submanifold (i.e., reachable from within the selected geodesic movement synergy) can be compared with the visual representation of the task space. The comparison can reinforce or suppress selection of the synergy symbol \((c, e_1, \ldots, e_N)\). This provides a recursive reinforcement-learning procedure. Over time, through trial and error, imitation and coaching, the nervous system forms associations between visual representations of task space and the appropriate geodesic movement synergy compatible with that task space.

But what happens if a totally geodesic submanifold compatible with the nominated task space does not exist? Using the Riemannian manifold theory of ‘variations through geodesics and Jacobi vector fields’ (Lang, 1999; Lee, 1997) it can be shown that a smooth submanifold partly spanned by geodesic coordinate grid lines compatible with the nominated task space can always be constructed using the method given in Section 5.3 and Appendix A up to the point where vertical geodesic coordinate grid lines are generated. Vertical coordinate grid lines are then constructed by connecting points at equal metric distances \(x^1\) along the horizontal geodesic coordinate grid lines. Submanifolds compatible with the task that are not totally geodesic but are partly spanned by free-motion horizontal geodesic coordinate grid lines can always be generated in this way. If a totally geodesic submanifold compatible with the task space does not exist then the submanifold that deviates minimally from a totally geodesic one is the one that minimizes demand for muscular effort.

Such a mechanism for selecting movement synergies gives an important extension to our computational theory (AMT) of movement planning and control. The selected symbol \((c, e_1, \ldots, e_N)\) not only uniquely specifies the required geodesic movement synergy but it also uniquely specifies all of the other sensory–sensory and sensorimotor transformations held by AMT to be required within the nervous system to plan and execute movement trajectories within the selected movement synergy. This greatly increases the importance of the symbol \((c, e_1, \ldots, e_N)\). Selecting \((c, e_1, \ldots, e_N)\) becomes the lynch pin so to speak...
for switching quickly and smoothly from one movement synergeme to another and for presetting sensory and motor systems in readiness to execute planned movement trajectories within each synergeme.

9.6. Riemannian field theory of human movement

Apart from providing an account of brain function, the Riemannian geometry theory of geodesic movement synergies developed here provides a mathematical framework that can be used to develop a whole-body coordinated movement simulator. The main barrier to implementation of a whole-body simulator is the need to include the mass–inertia matrix of the human body as a metric tensor field \( f(c) \) over the configuration manifold \( (C, f) \). This requires development of a database that specifies the mass–inertia about every elemental movement in every possible configuration of the body, including jumping, standing, leaning, sitting, lying, etc. Much like a genome project in extent, the development of such a database would enable identification and simulation of the most energy efficient whole-body coordinated movements to achieve arbitrarily nominated task goals, a matter of importance in many applications in human movement science.

The configuration manifold and its various vector and tensor spaces proposed in this paper provide the groundwork for a Riemannian field theory of human movement. The vectors, tensors and visuospatial memories are seen as being stored in distributed memory networks with the configuration of the body acting as an access code for those memories. Thus the large amount of information that the nervous system requires to plan purposive, goal-oriented, minimum-effort, coordinated movements taking nonlinear dynamics and interactions with the environment into account is represented geometrically by vector and tensor fields over configuration and perceived visual space manifolds. Resolution of paradoxes in existing explanations of experimental observations and new insights into human motor behaviour become possible. For example, movements that include the influences of potential energy fields, external force fields, dissipative systems, time-dependent systems, stability and balance can be analyzed using methods of Riemannian geometry (see for example Bullo & Lewis, 2005). And with respect to the psychological and neurophysiological notions of sensory–spatial maps as well as cognitive models for phenomena such as visual perspective, optical flow, three-dimensional mental imagery, localization of sounds, speech motor control, haptic exploration, and navigation, in each case the geometric theory of mappings between Riemannian manifolds offers the opportunity for exploration and development.

Appendix A. Two-CDOF geodesic movement synergies

Unlike the one-dimensional case, constructing a totally geodesic two-dimensional submanifold is not straightforward. Integral flows of two arbitrarily chosen linearly independent vector fields on a Riemannian manifold are usually not commutative. That is, integral flow along one vector field followed by integral flow along another usually does not end up at the same point in the manifold as does integral flow along the same vector fields in the reverse order. Typically the integral flows do not even intersect, so they do not join up in an appropriate way to form a coordinate system for a two-dimensional submanifold. Particular linearly-independent vector fields with integral flows that commute and do form a coordinate system have to be found. We now describe one way in which this can be done.

We seek to construct a coordinate system for a two-dimensional totally geodesic submanifold in \((C, f)\) centred about a specified point \( c \in (C, f) \). Suppose we have selected two \( J \)-orthonormal vectors \( e_1 \) and \( e_2 \) in the tangent space \( T_C \) to provide initial conditions \((c, e_1)\) and \((c, e_2)\). We can then deploy these in the GTG (Fig. 1) along with the predetermined spray acceleration-deceleration field \( f_L(c, v) \) stored in \( TTC \), to generate two unit metric-speed geodesic trajectories \( \alpha(x^1) \) and \( \beta(x^2) \) in \((C, f)\) parametrized by metric-distances \( x^1 \) and \( x^2 \) measured from the point \( c \in (C, f) \). The \( \alpha(x^1) \) and \( \beta(x^2) \) so obtained form geodesic coordinate axes for the two-dimensional totally geodesic submanifold specified by \((c, e_1)\) and \((c, e_2)\). Let us call this submanifold \( S_{\text{II}} \).

While we have geodesic coordinate axes \( \alpha(x^1) \) and \( \beta(x^2) \) for \( S_{\text{II}} \) we do not yet have geodesic coordinate grid lines. As described by Lee (1997), for any embedded submanifold like \( S_{\text{II}} \) there exist coordinates \( (x^1, \ldots, x^{116}) \) for \((C, f)\) centred about \( c \in (C, f) \) known as slice coordinates such that \( (x^1, x^2) \) form local coordinates for the submanifold. The slice coordinates \( x^1 \) and \( x^2 \) correspond to the metric-distances along the unit metric-speed geodesic trajectories \( \alpha(x^1) \) and \( \beta(x^2) \) that provide the geodesic coordinate axes, as illustrated in Fig. A1, however, the slice coordinate grid lines are not geodesic pathways. The diffeomorphic map \( \Psi : S_{\text{II}} \rightarrow \mathbb{R}^2 \) in Fig. A2 forms a slice coordinate chart for the two-dimensional local submanifold \( S_{\text{II}} \) embedded in \((C, f)\) shown in Fig. A1. At each point \( p \in S_{\text{II}} \) the two-dimensional vector space \( T_pS_{\text{II}} \) tangent to \( S_{\text{II}} \) can be identified as the subspace of \( T_pC \) spanned by slice coordinate vectors \( \partial_{x^1}, \partial_{x^2} \). To illustrate with a simple example, think of a loaf of bread. There exist slice coordinates \( (x^1, x^2, x^3) \) on the loaf of bread such that there is a particular slice with local slice coordinates \( (x^1, x^2, x^3) \) and slice coordinate vectors \( (\partial_{x^1}, \partial_{x^2}) \). This geometry provides us with a coordinate system.

We now describe exactly how the subset of points in \((C, f)\) spanned by the geodesic coordinate axes \( \alpha(x^1) \) and \( \beta(x^2) \) and forming the two-dimensional submanifold \( S_{\text{II}} \) is determined. The selected \( J \)-orthonormal vectors \( e_1 \) and \( e_2 \) span a two-dimensional vector subspace (plane) \( II \) in \( T_C \) (as in Fig. A1). This plane \( II \) defines a family of unit metric-norm vectors \( \{ \partial/\partial r_i \} \) \( i = 1, 2, \ldots \) (including \( e_1 \) and \( e_2 \)) in the plane \( II \) pointing in every direction in \( II \). Use this family of unit metric-norm vectors \( \{ \partial/\partial r_i \} \) as initial conditions \((c, \{ \partial/\partial r_i \})\) in the GTG to generate a family of unit metric-speed radial geodesic
trajecories \{γ_i\} (including \(\alpha(x^1)\) and \(\beta(x^2)\)) emanating from \(c \in (C,J)\) and parametrized by radial metric-distances \(\{r_i\}\), as illustrated in Figs. A1 and A2. If \((C,J)\) is locally positively curved (like a sphere) it is possible for the radial geodesics \(\{γ_i\}\) to converge back on each other and cross, just as longitude lines emanating from the north pole of a sphere converge back on each other and cross at the south pole. If this happens, it destroys the smooth, one-to-one, onto, invertible (i.e., diffeomorphic) map (known as the exponential map) between initial conditions (i.e., points in \(T_C\) corresponding to the family of unit metric-norm vectors \(∂/∂x^1\) scaled by time \(t\)) and points in \((C,J)\) (\(S_{II}\) in Fig. A1) reached by the family of radial geodesics \(\{γ_i\}\) in time \(t\). Crossings of the radial geodesics prevent the formation of a coordinate chart. To stop this from happening, the maximum metric-distances \(\{r_{i,\text{max}}\}\) along the radial geodesic trajectories \(\{γ_i\}\) have to be limited (by limiting time of flow \(t\) or metric distance \(r\) along each radial trajectory) so that no radial geodesic trajectory crossings occur. This defines a local diffeomorphic exponential map between a neighbourhood of points centred about the origin in \(T_C\) (plane II in Fig. A1) and a neighbourhood of points in \((C,J)\) (\(S_{II}\) in Fig. A1) centred about the point \(c \in (C,J)\). The family of radial geodesic trajectories \(\{γ_i\}\) with limited maximal metric-distances \(\{r_{i,\text{max}}\}\) sweep out a local two-dimensional submanifold \(S_{II}\) centred about \(c \in (C,J)\) embedded in \((C,J)\) called the plane section determined by II. This is analogous to the family of longitude lines emanating from the north pole on a sphere sweeping out the two-dimensional surface of the sphere (as illustrated in Fig. A1). Consequently, the set of points in \((C,J)\) reached by the non-crossing radial geodesics \(\{γ_i\}\) define the submanifold \(S_{II}\).

Next we need to add the necessary and sufficient conditions that must be satisfied by the initial conditions \((c,e_1)\) and \((c,e_2)\) for the resulting submanifold to be totally geodesic. In order to do so we first describe some important properties possessed by all two-dimensional submanifolds constructed as above by means of radial geodesic trajectories \(\{γ_i\}\) emanating from a point \(c \in (C,J)\). The intrinsic (Gaussian) curvature \(K(e_1,e_2)\) of \(S_{II}\) at \(c \in (C,J)\) equals the sectional curvature \(\tilde{K}(e_1,e_2)\) of the ambient manifold \((C,J)\) associated with II at the point \(c \in (C,J)\). The sum of all the sectional curvatures \(\tilde{K}(e_i,e_j)\) associated with all combinations of pairs of J-orthonormal vectors \((e_1,e_2,...,e_{116})\) spanning \(T_C\) completely determines the curvature of \((C,J)\) at the point \(c \in (C,J)\).

If the above procedure for constructing \(S_{II}\) is extended to geodesic trajectories generated by all of the J-orthonormal vectors \((e_1,e_2,...,e_{116})\) spanning \(T_C\) then the open set of points in \((C,J)\) reached by all the non-crossing radial geodesics is called a normal neighbourhood of the point \(c \in (C,J)\) and the metric-distances \((x^1,x^2,...,x^{116})\) along the geodesic trajectories define normal coordinates about the point \(c \in (C,J)\). Normal neighbourhoods and normal coordinates play an important role in Riemannian geometry. The submanifold \(S_{II}\) can be thought of as a two-dimensional slice of a normal neighbourhood.

The procedure for constructing \(S_{II}\) described so far only partly satisfies the requirements for \(S_{II}\) to be totally geodesic. While coordinate axes \(\alpha(x^1)\) and \(\beta(x^2)\) are geodesic trajectories in \((C,J)\) and while every local radial geodesic trajectory \(\{γ_i\}\) in \((C,J)\) emanating from \(c \in (C,J)\) is, by construction, totally contained within \(S_{II}\) as required, this does not hold for other points \(p \in S_{II}\). It is only at \(c \in S_{II}\) that, by construction, all geodesic trajectories starting tangent to \(S_{II}\) remain in \(S_{II}\). For \(S_{II}\) to be totally geodesic all geodesic trajectories in \((C,J)\) starting tangent to \(S_{II}\) at any point \(p \in S_{II}\) have to be contained totally within \(S_{II}\) and the intrinsic curvature \(K\) of \(S_{II}\) at every point of \(S_{II}\) has to equal the sectional curvature \(\tilde{K}\) of the ambient manifold \((C,J)\) at every point of \(S_{II}\), not just at \(c \in S_{II}\). In other words, the submanifold \(S_{II}\) must not be curved relative to the ambient manifold.

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**Fig. A1.** Construction of a local two-dimensional submanifold \(S_{II}\) swept out by a family of unit metric-speed radial geodesic trajectories \(\{\gamma_i\}\) emanating from the point \(c \in (C,J)\). II is the plane in the tangent velocity vector space \(T_C\) at the point \(c \in (C,J)\) spanned by the selected J-orthonormal vectors \(e_1\) and \(e_2\) as described in the text. The radial geodesic trajectories \(\alpha(x^1)\) and \(\beta(x^2)\) are generated by the GTG using initial conditions \((c,e_1)\) and \((c,e_2)\) respectively. \(x^1\) and \(x^2\) are metric-distances measured from \(c = (0,0)\) along \(\alpha(x^1)\) and \(\beta(x^2)\) respectively. The other radial geodesic trajectories \(\{\gamma_i\}\) (grey) emanating from the point \(c \in (C,J)\) are generated by the GTG using all the other unit metric-length vectors \(∂/∂x^1\) pointing in every direction in the plane II. The metric-distances along the radial geodesic trajectories defining \(S_{II}\) are limited so that no geodesic crossings occur. See text for detail.
Illustration of slice coordinates \((x^1, x^2)\) on a two-dimensional submanifold (plane section) \(S_\Pi\) embedded about the point \(c\) in the configuration manifold \((C, J)\). The diffeomorphic map \(\Psi : S_\Pi \rightarrow \mathbb{R}^2\) between the two-dimensional submanifold \(S_\Pi\) and the two-dimensional Euclidean space \(\mathbb{R}^2\) defines a slice coordinate chart for \(S_\Pi\). The radial geodesic trajectories \(\alpha(x^1)\) and \(\beta(x^2)\) parametrized by metric-distances \(x^1\) and \(x^2\) respectively, form geodesic coordinate axes for \(S_\Pi\). The coordinate grid lines are not necessarily geodesic curves.

Illustration of geodesic coordinate grid lines constructed on \(S_\Pi\). The \(J\)-orthonormal vectors \(e_1\) and \(e_2\) at the point \(c\) in \((C, J)\) are parallel translated along the geodesic coordinate axes \(\alpha(x^1)\) and \(\beta(x^2)\). If the submanifold \(S_\Pi\) is totally geodesic, the parallel translated vectors \(P_1\) and \(P_2\) remain tangent to \(S_\Pi\) at all points \(x^1\) and \(x^2\) along \(\alpha(x^1)\) and \(\beta(x^2)\). At each point \(x^2\) along \(\beta(x^2)\) initial conditions \((x^2, P_2)\) are used in the GTG to generate a family of horizontal geodesic trajectories \(\{\xi_x\}\). Similarly, at each point \(x^1\) along \(\alpha(x^1)\) initial conditions \((x^1, P_1)\) are used in the GTG to generate a family of vertical geodesic trajectories \(\{\eta_x\}\). If \(S_\Pi\) is totally geodesic the two families of horizontal and vertical geodesic trajectories \(\{\xi_x\}\) and \(\{\eta_x\}\) respectively are contained in \(S_\Pi\) and therefore they intersect forming a system of geodesic coordinate grid lines on \(S_\Pi\). The diffeomorphic map \(\Gamma : S_\Pi \rightarrow \mathbb{R}^2\) forms a geodesic coordinate chart on \(S_\Pi\). As described in the text, a velocity vector \(v\) at any point in \(S_\Pi\) can be projected via parallel translation along geodesic coordinate grid lines on to the geodesic coordinate axes \(\alpha(x^1)\) and \(\beta(x^2)\).

\((C, J)\), it must be a simple two-dimensional patch of \((C, J)\) with the same curvature as the sectional curvature of \((C, J)\) at every point \(p \in S_\Pi\). For this to be so the \(J\)-orthonormal vectors \(e_1\) and \(e_2\) have to be chosen so that when they are parallel translated along any of the radial geodesic trajectories \(\{\gamma_y\}\) (including \(\alpha(x^1)\) and \(\beta(x^2)\)) they remain tangent to \(S_\Pi\). This can be tested by checking that the parallel translated vectors \(P_1\) and \(P_2\) (illustrated in Fig. A3) can be expressed as a linear combination of the slice coordinate vectors \(\partial_{\alpha^1}, \partial_{\alpha^2}\) described above at each \(p \in S_\Pi\) and that the covariant derivative \(\nabla_u v\) of any two vectors \(u\) and \(v\) tangent to \(S_\Pi\) at any \(p \in S_\Pi\) is also tangent to \(S_\Pi\) at \(p \in S_\Pi\). Equivalently, the so-called second fundamental form (i.e., the component of the covariant derivative \(\nabla_u v\) normal to the submanifold \(S_\Pi\)) must vanish at every \(p \in S_\Pi\). Only when these conditions are satisfied for every \(p \in S_\Pi\) can it be concluded that \(S_\Pi\) is a totally geodesic submanifold.

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A vector \(Y\) is said to be parallel translated along a trajectory \(x\) when the covariant derivative \(\nabla_x Y\) equals zero at every point along \(x\). Furthermore, from Eq. (4), \(\nabla_x (Y.Z) = (\nabla_x Y).Z + (Y. \nabla_x Z)\), it can be deduced that the metric inner product of any two vectors \(Y\) and \(Z\) parallel translated along a trajectory \(x\) remains constant at every point along \(x\) and consequently, the metric norms and relative angles between vectors parallel translated along a trajectory \(x\) remain unchanged along \(x\).
Finally we show that geodesic coordinate grid lines exist on $S_II$. When $S_II$ is totally geodesic, the vectors $P_1$ and $P_2$ parallel translated along the geodesic coordinate axes $\beta(x^2)$ and $\alpha(x^1)$ respectively, can be used as initial conditions for the GTG to generate horizontal and vertical geodesic coordinate grid lines on $S_II$, as illustrated in Fig. A3. Using these geodesic coordinate grid lines, any point $p = (x^1, x^2) \in S_II$ can be projected on to the geodesic coordinate axes $\alpha(x^1)$ and $\beta(x^2)$ at $(x^1, 0)$ and $(0, x^2)$, respectively. Similarly, any vector $v$ with metric norm $||v||_{x^1,x^2}$ tangent to $S_II$ at the point $p = (x^1, x^2) \in S_II$ at an angle $\theta$ relative to the horizontal geodesic coordinate grid line passing through the point $p \in S_II$ (illustrated in Fig. A3) can be projected as $||v||_{x^1,x^2} \sin \theta$ and $||v||_{x^1,x^2} \cos \theta$ on to the geodesic coordinate axes $\beta(x^2)$ and $\alpha(x^1)$, respectively.

Ability to project positions and velocities in the submanifold on to geodesic coordinate axes spanning the submanifold along geodesic coordinate grid lines in this way is a special property of totally geodesic submanifolds constructed as described above. Consequently, any movement trajectory in $S_II$ can be decomposed into its two CDOFs by projection on to the two geodesic coordinate axes $\alpha(x^1)$ and $\beta(x^2)$ and conversely, any movement trajectory with two CDOFs contained in $S_II$ can be constructed by independently generating trajectories along the two geodesic coordinate axes $\alpha(x^1)$ and $\beta(x^2)$ and then combining them by projecting back along geodesic coordinate grid lines. This is illustrated in Section 7.2.

References


